数学II　第14回　多重積分

2016年1月19日　　担当：佐藤 純

問題1 以下の多重積分を計算せよ.

(1-1) $\displaystyle\iint_D xy\mathrm{d}x\mathrm{d}y, \qquad D: 0 \le x \le 1,\ 0 \le y \le x$

$$
\begin{aligned}
\iint_D xy\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y xy \\
&= \int_0^1 \mathrm{d}x \left[\frac{1}{2}xy^2\right]_{y=0}^{y=x} \\
&= \int_0^1 \mathrm{d}x \frac{1}{2}x^3 \\
&= \left[\frac{1}{8}x^4\right]_{x=0}^{x=1} \\
&= \boxed{\frac{1}{8}}
\end{aligned}
$$

(1-2) $\displaystyle\iint_D y\mathrm{d}x\mathrm{d}y, \qquad D: 0 \le x \le 1,\ x^2 \le y \le x$

$$
\begin{aligned}
\iint_D xy\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_{x^2}^x \mathrm{d}y\ y \\
&= \int_0^1 \mathrm{d}x \left[\frac{1}{2}y^2\right]_{y=x^2}^{y=x} \\
&= \int_0^1 \mathrm{d}x \frac{1}{2}(x^2 - x^4) \\
&= \left[\frac{1}{6}x^3 - \frac{1}{10}x^5\right]_{x=0}^{x=1} \\
&= \boxed{\frac{1}{15}}
\end{aligned}
$$

(1-3) $\displaystyle\iint_D (x+y)^2\mathrm{d}x\mathrm{d}y, \qquad D: x \ge 0, y \ge 0,\ x+y \le 1$

領域 D は，3点 $(0,1)$, $(0,0)$, $(1,0)$ を頂点とする三角形の内部である.

$$
\begin{aligned}
D &= \{x \ge 0, y \ge 0,\ x+y \le 1\} \\
&= \{0 \le x \le 1,\ 0 \le y \le 1-x\},
\end{aligned}
$$

$$
\begin{aligned}
\iint_D (x+y)^2\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y(x+y)^2 \\
&= \int_0^1 \mathrm{d}x \left[\frac{1}{3}(x+y)^3\right]_{y=0}^{y=1-x} \\
&= \int_0^1 \mathrm{d}x \frac{1}{3}(1-x^3) \\
&= \frac{1}{3}\left[x - \frac{1}{4}x^4\right]_{x=0}^{x=1}
\end{aligned}
$$

$$= \boxed{\frac{1}{4}}$$

(1-4) $\displaystyle\iint_D y\mathrm{d}x\mathrm{d}y, \qquad D: y \geq 0,\ x^2+y^2 \leq 1$

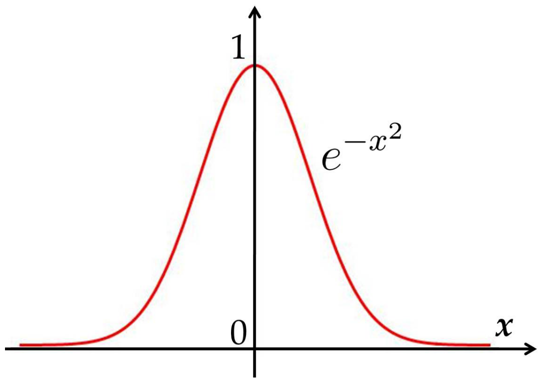
領域 D は，中心 $(0,0)$, 半径 1 の上半円の内部である．

$$\begin{aligned} D &= \{y \geq 0,\ x^2+y^2 \leq 1\} \\ &= \{-1 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2}\}, \end{aligned}$$

$$\begin{aligned} \iint_D y\mathrm{d}x\mathrm{d}y &= \int_{-1}^{1} \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y\ y \\ &= \int_{-1}^{1} \mathrm{d}x \left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{1-x^2}} \\ &= \int_{-1}^{1} \mathrm{d}x \frac{1}{2}(1-x^2) \\ &= \frac{1}{2}\left[x - \frac{1}{3}x^3\right]_{x=-1}^{x=1} \\ &= \boxed{\frac{2}{3}} \end{aligned}$$

問題 2

(2-1) e^{-x^2} のグラフの概形を書け．



(2-2) 定積分 $\displaystyle\int_0^\infty e^{-x^2}\mathrm{d}x$ を計算せよ．

$I = \displaystyle\int_0^\infty e^{-x^2}\mathrm{d}x$ とし，I^2 を計算すると，

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-x^2}\mathrm{d}x\right)\left(\int_0^\infty e^{-y^2}\mathrm{d}y\right) \\ &= \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\ e^{-(x^2+y^2)} \\ &= \int_0^\infty \mathrm{d}r \int_0^{\frac{\pi}{2}} \mathrm{d}\theta\ re^{-r^2} \\ &= \left(\int_0^\infty re^{-r^2}\mathrm{d}r\right)\left(\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left[e^{-r^2}\right]_0^\infty [\theta]_0^{\frac{\pi}{2}} \\
&= -\frac{1}{2}(0-1)\left(\frac{\pi}{2}-0\right) \\
&= \frac{\pi}{4}
\end{aligned}
$$

より，$I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$ を得る.

$$\boxed{\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}}$$

問題 3 定積分

$$I = \int_0^\infty \frac{\sin x}{x}\mathrm{d}x$$

を計算したい.

(3-1) 2 重積分

$$\int_0^\infty \int_0^\infty e^{-xy}\sin x\mathrm{d}x\mathrm{d}y$$

を，先に x，次に y で累次積分せよ.

$y \neq 0$ のとき，

$$
\begin{aligned}
A &= \int e^{-xy}\sin x\mathrm{d}x \\
&= \int \left(-\frac{1}{y}e^{-xy}\right)' \sin x\mathrm{d}x \\
&= -\frac{1}{y}e^{-xy}\sin x - \int \left(-\frac{1}{y}e^{-xy}\right)(\sin x)'\mathrm{d}x \\
&= -\frac{1}{y}e^{-xy}\sin x + \frac{1}{y}\int_0^\infty e^{-xy}\cos x\mathrm{d}x \\
&= -\frac{1}{y}e^{-xy}\sin x + \frac{1}{y}\int \left(-\frac{1}{y}e^{-xy}\right)' \cos x\mathrm{d}x \\
&= -\frac{1}{y}e^{-xy}\sin x - \frac{1}{y^2}e^{-xy}\cos x - \frac{1}{y}\int \left(-\frac{1}{y}e^{-xy}\right)(\cos x)'\mathrm{d}x \\
&= -\frac{1}{y}e^{-xy}\sin x - \frac{1}{y^2}e^{-xy}\cos x - \frac{1}{y^2}\int e^{-xy}\sin x\mathrm{d}x \\
&= -\frac{1}{y^2}e^{-xy}(y\sin x + \cos x) - \frac{1}{y^2}A
\end{aligned}
$$

より，

$$
\begin{aligned}
&A = -\frac{1}{y^2}e^{-xy}(y\sin x + \cos x) - \frac{1}{y^2}A, \\
&-y^2A = e^{-xy}(y\sin x + \cos x) + A, \\
&-(1+y^2)A = e^{-xy}(y\sin x + \cos x),
\end{aligned}
$$

$$A = -\frac{1}{1+y^2}e^{-xy}(y\sin x + \cos x)$$

となる．$y=0$ のときは $A = \int \sin x \mathrm{d}x = -\cos x$ となり，結局すべての y に対して $A = -\frac{1}{1+y^2}e^{-xy}(y\sin x + \cos x)$ が成り立つ．したがって，

$$\begin{aligned}
\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x e^{-xy}\sin x &= \int_0^\infty \mathrm{d}y \left[-\frac{1}{1+y^2}e^{-xy}(y\sin x + \cos x)\right]_{x=0}^{x=\infty} \\
&= \int_0^\infty \mathrm{d}y \frac{1}{1+y^2} \\
&= \left[\tan^{-1} y\right]_0^\infty \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}$$

を得る．

(3-2) 上と同じ 2 重積分を，先に y，次に x で累次積分せよ．

$$\begin{aligned}
\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y e^{-xy}\sin x &= \int_0^\infty \mathrm{d}x \sin x \int_0^\infty \mathrm{d}y e^{-xy} \\
&= \int_0^\infty \mathrm{d}x \sin x \left[-\frac{1}{x}e^{-xy}\right]_{y=0}^{y=\infty} \\
&= \boxed{\int_0^\infty \frac{\sin x}{x}\mathrm{d}x}
\end{aligned}$$

(3-3) これらの結果から定積分 I の値を求めよ．

積分の順序を入れ替えても値は変わらないので

$$\boxed{\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}}$$

を得る．