数学II　第13回　ベクトル解析

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問題1 スカラー場 f とベクトル場 $\overrightarrow{g}$ が,

$$f(x,y,z) = x^2 + y^2 + z^2,$$
$$\overrightarrow{g}(x,y,z) = (x+y+z, xy+yz+zx, xyz)$$

で与えられているとき，以下の量を計算せよ.

$$\overrightarrow{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \Delta = \overrightarrow{\nabla}\cdot\overrightarrow{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(1-1) $\overrightarrow{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \boxed{(2x, 2y, 2z)}$

(1-2) $\overrightarrow{\nabla}\cdot\overrightarrow{g} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(g_x, g_y, g_z) = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} = \boxed{1 + x + z + xy}$

(1-3) $\overrightarrow{\nabla}\times\overrightarrow{g} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(g_x, g_y, g_z) = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}, \frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}, \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)$

$= \boxed{(xz - y - x, 1 - yz, y + z - 1)}$

(1-4) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x^2+y^2+z^2)\right) = \frac{\partial}{\partial x}2x = 2.$

同様に，$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 2$ なので,

$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + 2 = \boxed{6}$

問題2 位置ベクトルを

$$\vec{r} = (x, y, z)$$
$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

と書くとき，以下の量を計算せよ.

(2-1) $\overrightarrow{\nabla} r = \frac{\vec{r}}{r}$

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{1}{2}} \\
&= \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(x^2+y^2+z^2)' \\
&= \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}2x \\
&= x(x^2+y^2+z^2)^{-\frac{1}{2}} \\
&= \frac{x}{\sqrt{x^2+y^2+z^2}}
\end{aligned}$$

$$= \frac{x}{r}$$

同様に,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

以上より,

$$\begin{aligned} \vec{\nabla} r &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) r \\ &= \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right) \\ &= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \\ &= \frac{1}{r}(x, y, z) \\ &= \boxed{\frac{\vec{r}}{r}} \end{aligned}$$

(2-2) $\vec{\nabla} \cdot \vec{r} = 3$

$$\begin{aligned} \vec{\nabla} \cdot \vec{r} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x, y, z) \\ &= \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z \\ &= 1 + 1 + 1 = \boxed{3} \end{aligned}$$

(2-3) $\vec{\nabla} \times \vec{r} = 0$

$$\begin{aligned} \vec{\nabla} \times \vec{r} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (x, y, z) \\ &= \left(\frac{\partial}{\partial y} z - \frac{\partial}{\partial z} y,\ \frac{\partial}{\partial z} x - \frac{\partial}{\partial x} z,\ \frac{\partial}{\partial x} y - \frac{\partial}{\partial y} x \right) \\ &= (0 - 0, 0 - 0, 0 - 0) = \boxed{0} \end{aligned}$$

(2-4) $\vec{\nabla} \frac{1}{r} = -\frac{\vec{r}}{r^3}$

$$\begin{aligned} \frac{\partial}{\partial x} \frac{1}{r} &= \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ &= -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} (x^2 + y^2 + z^2)' \\ &= -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} 2x \\ &= -x (x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ &= -\frac{x}{\left(\sqrt{x^2 + y^2 + z^2} \right)^3} \end{aligned}$$

$$= -\frac{x}{r^3}$$

同様に,

$$\frac{\partial}{\partial y}\frac{1}{r} = -\frac{y}{r^3}, \quad \frac{\partial}{\partial z}\frac{1}{r} = -\frac{z}{r^3}.$$

以上より,

$$\begin{aligned}
\vec{\nabla}\frac{1}{r} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} \\
&= \left(\frac{\partial}{\partial x}\frac{1}{r}, \frac{\partial}{\partial y}\frac{1}{r}, \frac{\partial}{\partial z}\frac{1}{r}\right) \\
&= \left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right) \\
&= -\frac{1}{r^3}(x, y, z) \\
&= \boxed{-\frac{\vec{r}}{r^3}}
\end{aligned}$$

(2-5) $\Delta\frac{1}{r} = 0$

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\frac{1}{r} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\frac{1}{r}\right) \\
&= -\frac{\partial}{\partial x}\frac{x}{r^3} \\
&= -\frac{\partial}{\partial x}x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\
&= -(x^2 + y^2 + z^2)^{-\frac{3}{2}} + \frac{3}{2}x(x^2 + y^2 + z^2)^{-\frac{5}{2}}2x \\
&= -\frac{1}{r^3} + \frac{3x^2}{r^5}
\end{aligned}$$

同様に,

$$\frac{\partial^2}{\partial y^2}\frac{1}{r} = -\frac{1}{r^3} + \frac{3y^2}{r^5}, \quad \frac{\partial^2}{\partial z^2}\frac{1}{r} = -\frac{1}{r^3} + \frac{3z^2}{r^5}.$$

以上より,

$$\begin{aligned}
\Delta\frac{1}{r} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{r} \\
&= \frac{\partial^2}{\partial x^2}\frac{1}{r} + \frac{\partial^2}{\partial y^2}\frac{1}{r} + \frac{\partial^2}{\partial z^2}\frac{1}{r} \\
&= -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} \\
&= -\frac{3}{r^3} + \frac{3r^2}{r^5} \\
&= \boxed{0}
\end{aligned}$$

問題 3 原点 $(0,0,0)$ に電荷 q があるときの電位は

$$\phi(x,y,z) = \frac{q}{4\pi\epsilon_0 r}$$

で与えられる．ただし，$r = \sqrt{x^2+y^2+z^2}$ である．

(3-1) 電場

$$\vec{E} = -\vec{\nabla}\phi$$

を計算せよ．

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}\phi \\
&= -\vec{\nabla}\left(\frac{q}{4\pi\epsilon_0 r}\right) \\
&= -\frac{q}{4\pi\epsilon_0}\left(\vec{\nabla}\frac{1}{r}\right)
\end{aligned}$$

ここで，

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{1}{r} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} \\
&= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(x^2+y^2+z^2)' \\
&= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2x \\
&= -x(x^2+y^2+z^2)^{-\frac{3}{2}} \\
&= -\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3} \\
&= -\frac{x}{r^3}
\end{aligned}$$

同様に，

$$\frac{\partial}{\partial y}\frac{1}{r} = -\frac{y}{r^3}, \quad \frac{\partial}{\partial z}\frac{1}{r} = -\frac{z}{r^3}.$$

以上より，

$$\begin{aligned}
\vec{\nabla}\frac{1}{r} &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\frac{1}{r} \\
&= \left(\frac{\partial}{\partial x}\frac{1}{r},\frac{\partial}{\partial y}\frac{1}{r},\frac{\partial}{\partial z}\frac{1}{r}\right) \\
&= \left(-\frac{x}{r^3},-\frac{y}{r^3},-\frac{z}{r^3}\right) \\
&= -\frac{1}{r^3}(x,y,z) \\
&= -\frac{\vec{r}}{r^3}.
\end{aligned}$$

したがって，

$$\begin{aligned}
\vec{E} &= -\frac{q}{4\pi\epsilon_0}\left(-\frac{\vec{r}}{r^3}\right) \\
&= \boxed{\frac{q}{4\pi\epsilon_0 r^3}\vec{r}}
\end{aligned}$$

(3-2) $\vec{\nabla} \cdot \vec{E} = 0$ を示せ.

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \left(\frac{q}{4\pi\epsilon_0 r^3} \vec{r} \right) = \frac{q}{4\pi\epsilon_0} \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right)$$

ここで,

$$\begin{aligned} \frac{\partial}{\partial x} \frac{x}{r^3} &= \frac{\partial}{\partial x} x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ &= (x^2+y^2+z^2)^{-\frac{3}{2}} - \frac{3}{2} x (x^2+y^2+z^2)^{-\frac{5}{2}} 2x \\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} \end{aligned}$$

同様に,

$$\frac{\partial}{\partial y} \frac{y}{r^3} = \frac{1}{r^3} - \frac{3y^2}{r^5}, \quad \frac{\partial}{\partial z} \frac{z}{r^3} = \frac{1}{r^3} - \frac{3z^2}{r^5}.$$

以上より,

$$\begin{aligned} \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right) \\ &= \frac{\partial}{\partial x} \frac{x}{r^3} + \frac{\partial}{\partial y} \frac{y}{r^3} + \frac{\partial}{\partial z} \frac{z}{r^3} \\ &= \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} \\ &= \frac{3}{r^3} - \frac{3r^2}{r^5} \\ &= 0. \end{aligned}$$

したがって,

$$\vec{\nabla} \cdot \vec{E} = 0$$

(3-3) $\vec{\nabla} \times \vec{E} = 0$ を示せ.

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= \vec{\nabla} \times \left(\frac{q}{4\pi\epsilon_0 r^3} \vec{r} \right) \\ &= \frac{q}{4\pi\epsilon_0} \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right) \\ &= \frac{q}{4\pi\epsilon_0} \left\{ \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right) \right\} \\ &= \frac{q}{4\pi\epsilon_0} \left(\frac{\partial}{\partial y} \frac{z}{r^3} - \frac{\partial}{\partial z} \frac{y}{r^3}, \frac{\partial}{\partial z} \frac{x}{r^3} - \frac{\partial}{\partial x} \frac{z}{r^3}, \frac{\partial}{\partial x} \frac{y}{r^3} - \frac{\partial}{\partial y} \frac{x}{r^3} \right) \\ &= \frac{q}{4\pi\epsilon_0} \left(z \frac{\partial}{\partial y} \frac{1}{r^3} - y \frac{\partial}{\partial z} \frac{1}{r^3}, x \frac{\partial}{\partial z} \frac{1}{r^3} - z \frac{\partial}{\partial x} \frac{1}{r^3}, y \frac{\partial}{\partial x} \frac{1}{r^3} - x \frac{\partial}{\partial y} \frac{1}{r^3} \right) \end{aligned}$$

ここで,

$$\frac{\partial}{\partial x} \frac{1}{r^3} = \frac{\partial}{\partial x} (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$
\begin{aligned}
&= -\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}2x \\
&= -\frac{3x}{r^5}
\end{aligned}
$$

同様に,

$$
\frac{\partial^2}{\partial y^2}\frac{1}{r} = -\frac{3y}{r^5}, \quad \frac{\partial^2}{\partial z^2}\frac{1}{r} = -\frac{3z}{r^5}.
$$

これを代入すれば直ちに $\overrightarrow{\nabla} \times \overrightarrow{E} = 0$ を得る.

別解：

$\overrightarrow{\nabla} \times \overrightarrow{E} = -\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla}\phi\right) = 0$

ただし，下の問題 (4-1) の結果を使った.

問題 4 以下の式を示せ

(4-1) $\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} f\right) = 0$

$$
\begin{aligned}
\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} f\right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
&= \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \\
&= 0.
\end{aligned}
$$

(4-2) $\overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{f}\right) = 0$

$$
\begin{aligned}
\overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{f}\right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \\
&= \left(\frac{\partial^2 f_z}{\partial x \partial y} - \frac{\partial^2 f_y}{\partial x \partial z}\right) + \left(\frac{\partial^2 f_x}{\partial y \partial z} - \frac{\partial^2 f_z}{\partial y \partial x}\right) + \left(\frac{\partial^2 f_y}{\partial z \partial x} - \frac{\partial^2 f_x}{\partial z \partial y}\right) \\
&= 0.
\end{aligned}
$$

(4-3) $\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{f}\right) = \overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \overrightarrow{f}\right) - \Delta \overrightarrow{f}$

$$
\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{f}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)
$$

$$
\begin{aligned}
&= \left(\frac{\partial}{\partial y} \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right), \right. \\
&\quad \frac{\partial}{\partial z} \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right), \\
&\quad \left. \frac{\partial}{\partial x} \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) \right) \\
&= \left(\frac{\partial^2 f_y}{\partial y \partial x} - \frac{\partial^2 f_x}{\partial y^2} - \frac{\partial^2 f_x}{\partial z^2} + \frac{\partial^2 f_z}{\partial z \partial x}, \right. \\
&\quad \frac{\partial^2 f_z}{\partial z \partial y} - \frac{\partial^2 f_y}{\partial z^2} - \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_x}{\partial x \partial y}, \\
&\quad \left. \frac{\partial^2 f_x}{\partial x \partial z} - \frac{\partial^2 f_z}{\partial x^2} - \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_y}{\partial y \partial z}, \right) \\
&= \left(\frac{\partial^2 f_y}{\partial y \partial x} + \frac{\partial^2 f_z}{\partial z \partial x} + \frac{\partial^2 f_x}{\partial x^2} - \frac{\partial^2 f_x}{\partial x^2} - \frac{\partial^2 f_x}{\partial y^2} - \frac{\partial^2 f_x}{\partial z^2}, \right. \\
&\quad \frac{\partial^2 f_z}{\partial z \partial y} + \frac{\partial^2 f_x}{\partial x \partial y} + \frac{\partial^2 f_y}{\partial y^2} - \frac{\partial^2 f_y}{\partial y^2} - \frac{\partial^2 f_y}{\partial z^2} - \frac{\partial^2 f_y}{\partial x^2}, \\
&\quad \left. \frac{\partial^2 f_x}{\partial x \partial z} + \frac{\partial^2 f_y}{\partial y \partial z} + \frac{\partial^2 f_z}{\partial z^2} - \frac{\partial^2 f_z}{\partial z^2} - \frac{\partial^2 f_z}{\partial x^2} - \frac{\partial^2 f_z}{\partial y^2}, \right) \\
&= \left(\frac{\partial}{\partial x} \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right) - \Delta f_x, \frac{\partial}{\partial y} \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right) - \Delta f_y, \frac{\partial}{\partial z} \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right) - \Delta f_z, \right) \\
&= \left(\frac{\partial}{\partial x} \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right), \frac{\partial}{\partial y} \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right), \frac{\partial}{\partial z} \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right) \right) - \left(\Delta f_x, \Delta f_y, \Delta f_z \right) \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right) - \Delta \left(f_x, f_y, f_z \right) \\
&= \overrightarrow{\nabla} \left(\overrightarrow{\nabla} \cdot \overrightarrow{f} \right) - \Delta \overrightarrow{f}.
\end{aligned}
$$