数学II　第11回　偏微分と全微分

2015年12月15日　　担当：佐藤 純

問題1 2変数 (x,y) の関数 $f(x,y)$ が以下の式で与えられているとき，1次偏導関数 $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ および全微分 $\mathrm{d}f$ を計算せよ．

(1-1) $f(x,y)=2x^2y-3xy^3$

$$\frac{\partial f}{\partial x}=\boxed{4xy-3y^3},\quad \frac{\partial f}{\partial y}=\boxed{2x^2-9xy^2},\quad \mathrm{d}f=\boxed{(4xy-3y^3)\mathrm{d}x+(2x^2-9xy^2)\mathrm{d}y}.$$

(1-2) $f(x,y)=\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2)^{1/2}=\frac{1}{2}(x^2+y^2)^{-1/2}(x^2+y^2)'=\frac{1}{2}(x^2+y^2)^{-1/2}2x=\frac{x}{\sqrt{x^2+y^2}}=\boxed{\frac{x}{f}},$$

同様に，$\dfrac{\partial f}{\partial y}=\boxed{\dfrac{y}{f}}$,　$\mathrm{d}f=\boxed{\dfrac{1}{f}(x\mathrm{d}x+y\mathrm{d}y)}$.

(1-3) $f(x,y)=e^{xy}$

$$\frac{\partial f}{\partial x}=\boxed{yf},\quad \frac{\partial f}{\partial y}=\boxed{xf},\quad \mathrm{d}f=\boxed{f(y\mathrm{d}x+x\mathrm{d}y)}.$$

(1-4) $f(x,y)=x\sin y$

$$\frac{\partial f}{\partial x}=\boxed{\sin y},\quad \frac{\partial f}{\partial y}=\boxed{x\cos y},\quad \mathrm{d}f=\boxed{\sin y\mathrm{d}x+x\cos y\mathrm{d}y}.$$

(1-5) $f(x,y)=\log(x^2+y^2)$

$$\frac{\partial f}{\partial x}=\boxed{\frac{2x}{x^2+y^2}},\quad \frac{\partial f}{\partial y}=\boxed{\frac{2y}{x^2+y^2}},\quad \mathrm{d}f=\boxed{\frac{2}{x^2+y^2}(x\mathrm{d}x+y\mathrm{d}y)}.$$

(1-6) $f(x,y)=\dfrac{x-y}{x+y}$

$$\frac{\partial f}{\partial x}=\boxed{\frac{2y}{(x+y)^2}},\quad \frac{\partial f}{\partial y}=\boxed{-\frac{2x}{(x+y)^2}},\quad \mathrm{d}f=\boxed{\frac{2}{(x+y)^2}(y\mathrm{d}x-x\mathrm{d}y)}.$$

問題2 2変数 (x,y) の関数 $f(x,y)$ が以下の式で与えられているとき，2次偏導関数 $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x\partial y}$, $\dfrac{\partial^2 f}{\partial y\partial x}$ を全て計算し，$\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}$ が成り立つことを確かめよ．

(2-1) $f(x,y)=2x^2y-3xy^3$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(4xy-3y^3)=\boxed{4y},$$
$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(2x^2-9xy^2)=\boxed{-18xy},$$
$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}(2x^2-9xy^2)=\boxed{4x-9y^2},$$
$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}(4xy-3y^3)=\boxed{4x-9y^2}.$$

(2-2) $f(x,y)=e^{xy}$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(ye^{xy}) = \boxed{y^2 e^{xy}},\\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}(xe^{xy}) = \boxed{x^2 e^{xy}},\\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}(xe^{xy}) = (x)'e^{xy} + x(e^{xy})' = e^{xy} + xye^{xy} = \boxed{(1+xy)e^{xy}},\\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(ye^{xy}) = \boxed{(1+xy)e^{xy}}.
\end{aligned}
$$

問題 3 2 次元 xy 座標 (x,y) と 2 次元極座標 (r,θ) の関係式は,

$$
\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}
$$

で与えられる．t を時刻とする物体の 2 次元運動を考える．

(3-1) r,θ を x,y の式で表せ.

$$
\begin{aligned}
&x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2\\
&\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta
\end{aligned}
$$

より,

$$
\boxed{\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\dfrac{y}{x} \end{cases}}
$$

(3-2) 物体の速度ベクトル $\vec{v} = \left(\dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}y}{\mathrm{d}t}\right)$ を $\dfrac{\mathrm{d}r}{\mathrm{d}t}, \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ を用いて表せ.

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial x}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial x}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \cos\theta\frac{\mathrm{d}r}{\mathrm{d}t} - r\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial y}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial y}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sin\theta\frac{\mathrm{d}r}{\mathrm{d}t} + r\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}
\end{aligned}
$$

$$
\boxed{\vec{v} = \left(\cos\theta\frac{\mathrm{d}r}{\mathrm{d}t} - r\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}, \sin\theta\frac{\mathrm{d}r}{\mathrm{d}t} + r\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)}
$$

(3-3) 角速度 ω の等速円運動のとき，$\dfrac{\mathrm{d}r}{\mathrm{d}t} = 0,\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$ となる．このとき,

$$
v = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = r\omega
$$

となることを示せ.

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = -r\omega\sin\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = r\omega\cos\theta
$$

より,

$$
\begin{aligned}
\boxed{v} &= \sqrt{(-r\omega\sin\theta)^2 + (r\omega\cos\theta)^2} \\
&= \sqrt{r^2\omega^2\sin^2\theta + r^2\omega^2\cos^2\theta} \\
&= \sqrt{r^2\omega^2(\sin^2\theta + \cos^2\theta)} \\
&= \sqrt{r^2\omega^2} \\
&= \boxed{r\omega}
\end{aligned}
$$

(3-4) 物体がポテンシャルエネルギー $U(x, y)$ の中を運動するとき，物体に働く力は

$$
\overrightarrow{F} = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)
$$

である．このとき，

$$
\overrightarrow{F} \cdot \vec{r} = -r\frac{\partial U}{\partial r}
$$

が成り立つことを示せ．ただし，位置ベクトルを $\vec{r} = (x, y)$ と書いた．

$$
\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-1/2}(x^2+y^2)' = \frac{1}{2}(x^2+y^2)^{-1/2}2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \\
\frac{\partial r}{\partial y} &= \frac{\partial}{\partial y}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-1/2}(x^2+y^2)' = \frac{1}{2}(x^2+y^2)^{-1/2}2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}
\end{aligned}
$$

$(\tan^{-1} x)' = \dfrac{1}{1+x^2}$ より,

$$
\begin{aligned}
\frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x}\tan^{-1}\frac{y}{x} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{y}{x}\right)' = \frac{1}{1+(\frac{y}{x})^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} \\
\frac{\partial \theta}{\partial y} &= \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{y}{x}\right)' = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{x}{r^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{x}{r}\frac{\partial U}{\partial r} - \frac{y}{r^2}\frac{\partial U}{\partial \theta} \\
\frac{\partial U}{\partial y} &= \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{y}{r}\frac{\partial U}{\partial r} + \frac{x}{r^2}\frac{\partial U}{\partial \theta}
\end{aligned}
$$

より,

$$
\begin{aligned}
\boxed{-\vec{F}\cdot\vec{r}} &= \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)\cdot(x, y) = x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \left(\frac{x}{r}\frac{\partial U}{\partial r} - \frac{y}{r^2}\frac{\partial U}{\partial \theta}\right)x + \left(\frac{y}{r}\frac{\partial U}{\partial r} + \frac{x}{r^2}\frac{\partial U}{\partial \theta}\right)y \\
&= \frac{x^2+y^2}{r}\frac{\partial U}{\partial r} + \frac{-yx+xy}{r^2}\frac{\partial U}{\partial \theta} = \boxed{r\frac{\partial U}{\partial r}}
\end{aligned}
$$

を得る．

(3-5) この力 $\overrightarrow{F}$ に逆らって物体を $\vec{r}$ から $\vec{r}+\mathrm{d}\vec{r}$ まで運ぶのに必要な仕事はポテンシャルの増分 $\mathrm{d}U$ に等しいことを示せ．ただし，$\mathrm{d}\vec{r} = (\mathrm{d}x, \mathrm{d}y)$, $\mathrm{d}U = U(x+\mathrm{d}x, y+\mathrm{d}y) - U(x, y)$ である．

$$
\boxed{\mathrm{d}W} = -\overrightarrow{F}\cdot\mathrm{d}\vec{r} = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)\cdot(\mathrm{d}x, \mathrm{d}y) = \frac{\partial U}{\partial x}\mathrm{d}x + \frac{\partial U}{\partial y}\mathrm{d}y = \boxed{\mathrm{d}U}
$$