数学 II　第 5 回　積分のまとめ

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問題 1 以下の不定積分を計算せよ.

(1-1) $\displaystyle\int \frac{\mathrm{d}x}{x^2-1} = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + C$

x^2-1 を因数分解すると，$x^2-1=(x+1)(x-1)$ なので，$\dfrac{1}{x^2-1}$ を部分分数分解して，

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{a}{x+1} + \frac{b}{x-1}$$

という形に書く．ここで，a, b は適当においた定数で，この式が成り立つようにこれから決める．両辺に $(x+1)(x-1)$ をかけると，

$$1 = a(x-1) + b(x+1)$$

を得る．これに $x=1$ を代入すると，$1=2b$ なので $b=1/2$ となり，$x=-1$ を代入すると，$1=-2a$ なので $a=-1/2$ となる．
よって，

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2-1} &= \int\left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\right)\mathrm{d}x = \frac{1}{2}\left(\int\frac{\mathrm{d}x}{x-1} - \int\frac{\mathrm{d}x}{x+1}\right)\\ &= \frac{1}{2}\left(\log(x-1) - \log(x+1)\right) + C\\ &= \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + C\end{aligned}$$

となる．

(1-2) $\displaystyle\int \frac{\mathrm{d}x}{x^2+1} = \tan^{-1}x + C$

$(\tan^{-1}x)' = \dfrac{1}{x^2+1}$ なので直ちに上式を得る．

これは公式そのものなので解答だけでよいが，参考までに，以下のような "計算" も可能であることを示す

$$\begin{aligned}y = \int\frac{\mathrm{d}x}{x^2+1} &= \frac{1}{2\mathrm{i}}\int\left(\frac{1}{x-\mathrm{i}} - \frac{1}{x+\mathrm{i}}\right)\mathrm{d}x\\ &= \frac{1}{2\mathrm{i}}\log\frac{x-\mathrm{i}}{x+\mathrm{i}} + C\end{aligned}$$

より，

$$\begin{aligned}&2\mathrm{i}(y-C) = \log\frac{x-\mathrm{i}}{x+\mathrm{i}},\\ &e^{2\mathrm{i}(y-C)} = \frac{x-\mathrm{i}}{x+\mathrm{i}},\\ &x = \mathrm{i}\frac{1+e^{2\mathrm{i}(y-C)}}{1-e^{2\mathrm{i}(y-C)}} = \mathrm{i}\frac{e^{-\mathrm{i}(y-C)}+e^{\mathrm{i}(y-C)}}{e^{-\mathrm{i}(y-C)}-e^{\mathrm{i}(y-C)}} = \mathrm{i}\frac{2\cos(y-C)}{-2\mathrm{i}\sin(y-C)} = -\frac{\cos(y-C)}{\sin(y-C)}\end{aligned}$$

を得る．ここで，$C' = C + \dfrac{\pi}{2}$ とすると，

$$x = -\frac{\cos(y-C'+\frac{\pi}{2})}{\sin(y-C'+\frac{\pi}{2})} = -\frac{-\sin(y-C')}{\cos(y-C')} = \tan(y-C'),$$

$$\tan^{-1} x = y - C'$$

より，$y = \tan^{-1} x + C'$ を得る．

(1-3) $\displaystyle\int \frac{\mathrm{d}x}{x^3+1} = \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$x^3+1 = (x+1)(x^2-x+1)$ と因数分解されるので，$\dfrac{1}{x^3+1}$ を部分分数分解して，

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$$

という形に書く．ここで，a, b, c は適当においた定数で，この式が成り立つようにこれから決める．両辺に $(x+1)(x^2-x+1)$ をかけると，

$$1 = a(x^2-x+1) + (x+1)(bx+c)$$

を得る．これに $x = -1$ を代入すると，$1 = 3a$ なので $a = 1/3$ となり，$x = 0$ を代入すると，$1 = a + c$ なので $c = 2/3$ となり，$x = 1$ を代入すると，$1 = a + 2(b+c)$ なので $3 = 3a + 2(3b+3c) = 1 + 2(3b+2) = 6b + 5$ より，$b = -1/3$ となる．

よって，

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^3+1} &= \frac{1}{3}\int \frac{\mathrm{d}x}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\mathrm{d}x \\
&= \frac{1}{3}\log(x+1) - \frac{1}{3}\int \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1}\mathrm{d}x \\
&= \frac{1}{3}\log(x+1) - \frac{1}{6}\int \frac{(x^2-x+1)'}{x^2-x+1}\mathrm{d}x + \frac{1}{2}\int \frac{\mathrm{d}x}{x^2-x+1} \\
&= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int \frac{\mathrm{d}x}{(x-\frac{1}{2})^2 + \frac{3}{4}}
\end{aligned}$$

となる．

ここで，$(x-\frac{1}{2})^2 = \frac{3}{4}t^2$ となるように変数を x から t に置換すると，

$x = \frac{\sqrt{3}}{2}t + \frac{1}{2}$, $t = \frac{2x-1}{\sqrt{3}}$, $\mathrm{d}x = \frac{\sqrt{3}}{2}\mathrm{d}t$ なので，

$$\begin{aligned}
\frac{1}{2}\int \frac{\mathrm{d}x}{(x-\frac{1}{2})^2+\frac{3}{4}} &= \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\mathrm{d}t}{\frac{3}{4}t^2 + \frac{3}{4}} = \frac{1}{\sqrt{3}}\int \frac{\mathrm{d}t}{t^2+1} = \frac{1}{\sqrt{3}}\tan^{-1} t + C \\
&= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C
\end{aligned}$$

となり，結局

$$\int \frac{\mathrm{d}x}{x^3+1} = \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

を得る．

(1-4) $\displaystyle\int \frac{\mathrm{d}x}{\sin x} = \log\left(\tan\frac{x}{2}\right) + C$

三角関数の有理関数の積分は置換 $t = \tan\frac{x}{2}$ によって t の有理関数の積分に帰着し，必ず可積分となる．

$1 + t^2 = 1 + \tan^2\frac{x}{2} = 1 + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1}{\cos^2\frac{x}{2}}$ であるが，

ここで，$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ において $\alpha = \beta = \frac{x}{2}$ とおくと，

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1$ より $\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$ なので，

$1 + t^2 = \frac{2}{1+\cos x}$, $(1+t^2)(1+\cos x) = 2$, $(1+t^2)\cos x = 2 - (1+t^2) = 1 - t^2$ より，

$\cos x = \frac{1-t^2}{1+t^2}$ を得る．
また，$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ において $\alpha = \beta = \frac{x}{2}$ とおくと，
$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{1+t^2}$ を得る．
また，$\frac{\mathrm{d}t}{\mathrm{d}x} = (\tan\frac{x}{2})' = \frac{1}{2}\frac{1}{\cos^2\frac{x}{2}} = \frac{1+t^2}{2}$ より，$\mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}$ を得る．
以上をまとめると，

$$\cos x = \frac{1-t^2}{1+t^2}, \qquad \sin x = \frac{2t}{1+t^2}, \qquad \mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}$$

となる．

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{1+t^2}{2t}\frac{2\mathrm{d}t}{1+t^2} = \int \frac{\mathrm{d}t}{t} = \log t + C = \log\left(\tan\frac{x}{2}\right) + C$$

別解：

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\sin x}{\sin^2 x}\mathrm{d}x = \int \frac{(-\cos x)'}{1-\cos^2 x}\mathrm{d}x$$

なので，$t = \cos x$ とおけば，$\mathrm{d}t = -\sin x \mathrm{d}x$ より，

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sin x} &= \int \frac{\mathrm{d}t}{t^2-1} = \frac{1}{2}\log\left(\frac{t-1}{t+1}\right) + C = \frac{1}{2}\log\left(\frac{\cos x - 1}{\cos x + 1}\right) + C \\ &= \frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right) + C = \frac{1}{2}\log\left(\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) + C = \frac{1}{2}\log\left(\tan^2\frac{x}{2}\right) + C \\ &= \log\left(\tan^2\frac{x}{2}\right)^{\frac{1}{2}} + C = \log\left(\tan\frac{x}{2}\right) + C\end{aligned}$$

(1-5) $\displaystyle\int \frac{\mathrm{d}x}{1+\sin x} = \boxed{-\frac{2}{1+\tan\frac{x}{2}} + C}$
置換 $t = \tan\frac{x}{2}$ によって

$$\int \frac{\mathrm{d}x}{1+\sin x} = \int \frac{1}{1+\frac{2t}{1+t^2}}\frac{2\mathrm{d}t}{1+t^2} = \int \frac{2\mathrm{d}t}{(1+t)^2} = -\frac{2}{1+t} + C = \boxed{-\frac{2}{1+\tan\frac{x}{2}} + C}$$

あるいは，もう少し式を綺麗にすると，$C = C' + 2$ として，

$$\begin{aligned}-\frac{2}{1+t} + C = -\frac{2}{1+t} + 2 + C' &= \frac{-2+2(1+t)}{1+t} + 2 + C' = \frac{2t}{1+t} + C' \\ &= \frac{2\tan\frac{x}{2}}{1+\tan\frac{x}{2}} + C' = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}} + C' \\ &= \frac{2\sin\frac{x}{2}(\cos\frac{x}{2}-\sin\frac{x}{2})}{(\cos\frac{x}{2}+\sin\frac{x}{2})(\cos\frac{x}{2}-\sin\frac{x}{2})} + C' \\ &= \frac{2\sin\frac{x}{2}\cos\frac{x}{2} - 2\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} + C' \\ &= \frac{\sin x - (1-\cos x)}{\cos x} + C' \\ &= \boxed{\tan x - \frac{1}{\cos x} + D}\end{aligned}$$

を得る．ただし，$C' + 1 = D$ とした．

(1-6) $\displaystyle\int\sqrt{x^2+1}\,\mathrm{d}x = \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\log(x+\sqrt{x^2+1})+C$

置換 $t = x+\sqrt{x^2+1}$ を用いて,

$$\begin{aligned}
&t-x=\sqrt{x^2+1},\\
&(t-x)^2 = x^2+1,\\
&t^2-2tx=1,\\
&x=\frac{t^2-1}{2t}=\frac{1}{2}\left(t-\frac{1}{t}\right),\\
&\sqrt{x^2+1}=t-x=t-\frac{1}{2}\left(t-\frac{1}{t}\right)=\frac{1}{2}\left(t+\frac{1}{t}\right),\\
&\mathrm{d}x=\frac{1}{2}\left(t-\frac{1}{t}\right)'\mathrm{d}t=\frac{1}{2}\left(1+\frac{1}{t^2}\right)\mathrm{d}t=\frac{1}{2t}\left(t+\frac{1}{t}\right)\mathrm{d}t
\end{aligned}$$

以上より,

$$\begin{aligned}
\int\sqrt{x^2+1}\,\mathrm{d}x &= \int\frac{1}{2}\left(t+\frac{1}{t}\right)\frac{1}{2t}\left(t+\frac{1}{t}\right)\mathrm{d}t\\
&=\int\frac{1}{4}\left(t+\frac{2}{t}+\frac{1}{t^3}\right)\mathrm{d}t\\
&=\frac{1}{8}t^2+\frac{1}{2}\log t-\frac{1}{8t^2}+C\\
&=\frac{1}{2}\times\frac{1}{2}\left(t-\frac{1}{t}\right)\times\frac{1}{2}\left(t+\frac{1}{t}\right)+\frac{1}{2}\log t+C\\
&=\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\log(x+\sqrt{x^2+1})+C
\end{aligned}$$

となる.

(1-7) $\displaystyle\int\sqrt{\frac{1-x}{x}}\,\mathrm{d}x = \sqrt{x(1-x)}-\tan^{-1}\sqrt{\frac{1-x}{x}}+C$

置換 $t=\sqrt{\frac{1-x}{x}}$ を用いて,

$$\begin{aligned}
&t^2=\frac{1-x}{x}=\frac{1}{x}-1,\\
&x=\frac{1}{1+t^2},\\
&\mathrm{d}x=\left(\frac{1}{1+t^2}\right)'\mathrm{d}t
\end{aligned}$$

より,

$$\begin{aligned}
\int\sqrt{\frac{1-x}{x}}\,\mathrm{d}x &= \int t\left(\frac{1}{1+t^2}\right)'\mathrm{d}t = t\,\frac{1}{1+t^2}-\int t'\,\frac{1}{1+t^2}\mathrm{d}t\\
&=\frac{t}{1+t^2}-\int\frac{\mathrm{d}t}{1+t^2}=\frac{t}{1+t^2}-\tan^{-1}t+C\\
&=\sqrt{x(1-x)}-\tan^{-1}\sqrt{\frac{1-x}{x}}+C
\end{aligned}$$

となる.

問題 2 正の実数 t に対し,ガンマ関数 $\Gamma(t)=\displaystyle\int_0^\infty x^{t-1}e^{-x}\mathrm{d}x$ を定義する.

(2-1) 漸化式 $\Gamma(t+1) = t\,\Gamma(t)$ を示せ.

$$\begin{aligned}
\Gamma(t+1) &= \int_0^\infty x^t e^{-x}\mathrm{d}x = \int_0^\infty x^t(-e^{-x})'\mathrm{d}x = -[x^t e^{-x}]_0^\infty + \int_0^\infty (x^t)'e^{-x}\mathrm{d}x \\
&= t\int_0^\infty x^{t-1}e^{-x}\mathrm{d}x = t\,\Gamma(t)
\end{aligned}$$

(2-2) 自然数 n に対し，$\Gamma(n) = (n-1)!$ を示せ.
ただし，$n! = 1\times 2\times 3\times\cdots\times(n-1)\times n$, $0! = 1$ である.

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-x}\mathrm{d}x = -[e^{-x}]_0^\infty = 1 = 0!, \\
\Gamma(n) &= (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) \\
&= \cdots = (n-1)(n-2)\cdots 2\cdot 1\cdot\Gamma(1) = (n-1)!
\end{aligned}$$

問題 3 正の実数 s, t に対し，ベータ関数 $B(s,t) = \displaystyle\int_0^1 x^{s-1}(1-x)^{t-1}\mathrm{d}x$ を定義する.

(3-1) $B(s,t) = B(t,s)$ を示せ.
$y = 1-x$ と置換すると，

$$\begin{aligned}
B(s,t) &= \int_0^1 x^{s-1}(1-x)^{t-1}\mathrm{d}x = \int_1^0 (1-y)^{s-1}y^{t-1}(-\mathrm{d}y) \\
&= \int_0^1 (1-y)^{s-1}y^{t-1}\mathrm{d}y = B(t,s)
\end{aligned}$$

(3-2) $B(s,t+1) = \dfrac{t}{s}B(s+1,t)$ を示せ.

$$\begin{aligned}
B(s,t+1) &= \int_0^1 x^{s-1}(1-x)^t\mathrm{d}x = \int_0^1 \left(\frac{1}{s}x^s\right)'(1-x)^t\mathrm{d}x \\
&= \left[\frac{1}{s}x^s(1-x)^t\right]_0^1 - \int_0^1 \frac{1}{s}x^s\{(1-x)^t\}'\mathrm{d}x = \frac{t}{s}\int_0^1 x^s(1-x)^{t-1}\mathrm{d}x \\
&= \frac{t}{s}B(s+1,t)
\end{aligned}$$

(3-3) 自然数 n, m に対し，$B(n,m) = \dfrac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$ を示せ.

$$\begin{aligned}
B(n,m) &= \frac{m-1}{n}B(n+1,m-1) = \left(\frac{m-1}{n}\frac{m-2}{n+1}\right)B(n+2,m-2) \\
&= \cdots = \left(\frac{m-1}{n}\frac{m-2}{n+1}\cdots\frac{1}{n+m-2}\right)B(n+m-1,1) \\
&= \left(\frac{m-1}{n}\frac{m-2}{n+1}\cdots\frac{1}{n+m-2}\right)\int_0^1 x^{n+m-2}\mathrm{d}x \\
&= \left(\frac{m-1}{n}\frac{m-2}{n+1}\cdots\frac{1}{n+m-2}\right)\left[\frac{1}{n+m-1}x^{n+m-1}\right]_0^1 \\
&= \frac{m-1}{n}\frac{m-2}{n+1}\cdots\frac{1}{n+m-2}\frac{1}{n+m-1}
\end{aligned}$$

$$= \frac{(m-1)(m-2)\cdots 2\cdot 1}{n(n+1)(n+2)\cdots(n+m-2)(n+m-1)}$$
$$= \frac{(m-1)!}{n(n+1)(n+2)\cdots(n+m-2)(n+m-1)} \times \frac{(n-1)!}{(n-1)!}$$
$$= \frac{(n-1)!(m-1)!}{(n+m-1)!} = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

(3-4) 自然数 n, m に対し，$\displaystyle\int_\alpha^\beta (x-\alpha)^n(\beta-x)^m \mathrm{d}x = (\beta-\alpha)^{n+m+1}\frac{n!m!}{(n+m+1)!}$ を示せ.

$x = \alpha + t(\beta-\alpha)$ と置換すれば，$\begin{array}{c|c} x & \alpha \to \beta \\ \hline t & 0 \to 1 \end{array}$ および $\beta - x = (\beta-\alpha)(1-t)$ より，

$$\begin{aligned}
\int_\alpha^\beta (x-\alpha)^n(\beta-x)^m \mathrm{d}x &= \int_0^1 \{t(\beta-\alpha)\}^n\{(\beta-\alpha)(1-t)\}^m(\beta-\alpha)\mathrm{d}t \\
&= (\beta-\alpha)^{n+m+1}\int_0^1 t^n(1-t)^m \mathrm{d}t \\
&= (\beta-\alpha)^{n+m+1} B(n+1, m+1) \\
&= (\beta-\alpha)^{n+m+1}\frac{n!m!}{(n+m+1)!}
\end{aligned}$$

を得る.

(3-5) 定積分 $\displaystyle\int_\alpha^\beta (x-\alpha)(\beta-x)\mathrm{d}x, \int_\alpha^\beta (x-\alpha)(\beta-x)^2\mathrm{d}x, \int_\alpha^\beta (x-\alpha)^2(\beta-x)^2\mathrm{d}x$ を計算せよ.

$$\begin{aligned}
\int_\alpha^\beta (x-\alpha)(\beta-x)\mathrm{d}x &= (\beta-\alpha)^3\frac{1!1!}{3!} = \frac{1}{6}(\beta-\alpha)^3, \\
\int_\alpha^\beta (x-\alpha)(\beta-x)^2\mathrm{d}x &= (\beta-\alpha)^4\frac{1!2!}{4!} = \frac{1}{12}(\beta-\alpha)^4, \\
\int_\alpha^\beta (x-\alpha)^2(\beta-x)^2\mathrm{d}x &= (\beta-\alpha)^5\frac{2!2!}{3!} = \frac{1}{30}(\beta-\alpha)^5.
\end{aligned}$$

問題 4 以下の積分を計算せよ.

(4-1) 定積分 $\displaystyle\int_1^{e^\pi} \sin(\log x)\mathrm{d}x$ の値を求めよ.

計算が優雅で労力が少ない順に，3 つ解法を示す.

(解法 1)：オイラーの公式

オイラーの公式 $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ より，$\sin\theta = \mathrm{Im}e^{\mathrm{i}\theta}$ なので，

$$\begin{aligned}
\int_1^{e^\pi} \sin(\log x)\mathrm{d}x &= \mathrm{Im}\int_1^{e^\pi} e^{\mathrm{i}\log x}\mathrm{d}x \\
&= \mathrm{Im}\int_1^{e^\pi} x^{\mathrm{i}}\mathrm{d}x \\
&= \mathrm{Im}\left[\frac{1}{\mathrm{i}+1}x^{\mathrm{i}+1}\right]_1^{e^\pi}
\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{Im}\frac{1}{\mathrm{i}+1}(e^{\pi}+1) \\
&= \frac{e^{\pi}+1}{2}
\end{aligned}
$$

(解法 2)：部分積分

$$
\begin{aligned}
I = \int_1^{e^{\pi}} \sin(\log x)\mathrm{d}x &= \int_1^{e^{\pi}} (x)' \sin(\log x)\mathrm{d}x \\
&= [x\sin(\log x)]_1^{e^{\pi}} - \int_1^{e^{\pi}} x\{\sin(\log x)\}'\mathrm{d}x \\
&= (e^{\pi}\sin\pi - \sin 0) - \int_1^{e^{\pi}} x\{\cos(\log x)\}(\log x)'\mathrm{d}x \\
&= (0-0) - \int_1^{e^{\pi}} x\{\cos(\log x)\}\frac{1}{x}\mathrm{d}x \\
&= -\int_1^{e^{\pi}} \cos(\log x)\mathrm{d}x \\
&= -\int_1^{e^{\pi}} (x)'\cos(\log x)\mathrm{d}x \\
&= -[x\cos(\log x)]_1^{e^{\pi}} + \int_1^{e^{\pi}} \{\cos(\log x)\}'\mathrm{d}x \\
&= -(e^{\pi}\cos\pi - \cos 0) - \int_1^{e^{\pi}} \sin(\log x)\mathrm{d}x \\
&= e^{\pi} + 1 - I
\end{aligned}
$$

より，$I = \dfrac{e^{\pi}+1}{2}$ を得る．

(解法 3)：置換積分

$t = \log x$ とおくと，$\begin{array}{c|c} x & 1 \to e^{\pi} \\ \hline t & 0 \to \pi \end{array}$ であり，$\mathrm{d}t = \dfrac{\mathrm{d}x}{x}$ より $\mathrm{d}x = x\mathrm{d}t = e^t\mathrm{d}t$ なので，

$$
\begin{aligned}
I = \int_1^{e^{\pi}} \sin(\log x)\mathrm{d}x &= \int_0^{\pi} e^t \sin t\mathrm{d}t \\
&= \int_0^{\pi} (e^t)'\sin t\mathrm{d}t \\
&= [e^t \sin t]_0^{\pi} - \int_0^{\pi} e^t\cos t\mathrm{d}t \\
&= -[e^t\cos t]_0^{\pi} - \int_0^{\pi} e^t \sin t\mathrm{d}t \\
&= e^{\pi} + 1 - I
\end{aligned}
$$

より，$I = \dfrac{e^{\pi}+1}{2}$ を得る．

(4-2) 定積分に読み替えることによって，極限値

$$
\lim_{n\to\infty} \frac{1}{n\sqrt{n}} \sum_{k=1}^{n} \sqrt{k}
$$

を計算せよ．

$$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\sqrt{k}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}=\int_0^1\sqrt{x}\mathrm{d}x=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1=\frac{2}{3}.$$

(4-3) 置換 $x=\tan\theta$ を用いて，定積分

$$\int_0^1\frac{\log(x+1)}{x^2+1}\mathrm{d}x$$

を計算せよ．

$x^2+1=\dfrac{1}{\cos^2\theta}$, $\mathrm{d}x=\dfrac{\mathrm{d}\theta}{\cos^2\theta}=(x^2+1)\mathrm{d}\theta$,

$$\begin{aligned}\int_0^1\frac{\log(x+1)}{x^2+1}\mathrm{d}x&=\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{4}}\log\left(1+\frac{\sin\theta}{\cos\theta}\right)\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{4}}\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right)\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{4}}\left\{\log(\cos\theta+\sin\theta)-\log\cos\theta\right\}\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{4}}\left(\log\left[\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right]-\log\cos\theta\right)\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{4}}\left(\log\sqrt{2}\right)\mathrm{d}\theta+\int_0^{\frac{\pi}{4}}\log\left[\cos\left(\theta-\frac{\pi}{4}\right)\right]\mathrm{d}\theta-\int_0^{\frac{\pi}{4}}\log\cos\theta\mathrm{d}\theta.\end{aligned}$$

ここで，$\theta'=\frac{\pi}{4}-\theta$ とすると，

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\log\left[\cos\left(\theta-\frac{\pi}{4}\right)\right]\mathrm{d}\theta&=\int_{\frac{\pi}{4}}^{0}\log\left[\cos(-\theta')\right](-1)\mathrm{d}\theta'\\&=\int_0^{\frac{\pi}{4}}\log\cos\theta'\mathrm{d}\theta'\end{aligned}$$

なので，

$$\int_0^1\frac{\log(x+1)}{x^2+1}\mathrm{d}x=\int_0^{\frac{\pi}{4}}\left(\log\sqrt{2}\right)\mathrm{d}\theta=\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\log 2\right)\mathrm{d}\theta=\left(\frac{1}{2}\log 2\right)\frac{\pi}{4}=\frac{\pi}{8}\log 2.$$