数学II　第2回　不定積分・定積分

2015年10月6日　担当：佐藤 純

問題1 以下の不定積分を計算せよ．

(1-1) $\displaystyle\int (x^2+2x+3)\mathrm{d}x = \frac{1}{3}x^3 + x^2 + 3x + C$

(1-2) $\displaystyle\int \frac{1}{x^3}\mathrm{d}x = -\frac{1}{2x^2} + C$

(1-3) $\displaystyle\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\mathrm{d}x = \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + C$

(1-4) $\displaystyle\int e^{-2x}\,\mathrm{d}x = -\frac{1}{2}e^{-2x} + C$

(1-5) 省略したやり方：

$$\begin{aligned}\int (3x^2-2x)(x^3-x^2+1)^3\,\mathrm{d}x &= \int (x^3-x^2+1)'(x^3-x^2+1)^3\,\mathrm{d}x \\ &= \frac{1}{4}(x^3-x^2+1)^4 + C\end{aligned}$$

置換積分をきちんと書くと：

$t = x^3 - x^2 + 1$ とおく．$\dfrac{\mathrm{d}t}{\mathrm{d}x} = 3x^2 - 2x$ より，$\mathrm{d}t = (3x^2-2x)\mathrm{d}x$ なので，

$$\begin{aligned}\int (3x^2-2x)(x^3-x^2+1)^3\,\mathrm{d}x &= \int t^3\mathrm{d}t \\ &= \frac{1}{4}t^4 + C \\ &= \frac{1}{4}(x^3-x^2+1)^4 + C\end{aligned}$$

(1-6) $\displaystyle\int \frac{3x^2-2x}{x^3-x^2+1}\,\mathrm{d}x$

省略したやり方：

$$\begin{aligned}\int \frac{3x^2-2x}{x^3-x^2+1}\,\mathrm{d}x &= \int \frac{(x^3-x^2+1)'}{(x^3-x^2+1)}\,\mathrm{d}x \\ &= \log|x^3-x^2+1| + C\end{aligned}$$

置換積分をきちんと書くと：

$t = x^3 - x^2 + 1$ とおく．$\dfrac{\mathrm{d}t}{\mathrm{d}x} = 3x^2 - 2x$ より，$\mathrm{d}t = (3x^2-2x)\mathrm{d}x$ なので，

$$\begin{aligned}\int \frac{3x^2-2x}{x^3-x^2+1}\,\mathrm{d}x &= \int \frac{\mathrm{d}t}{t} \\ &= \log|t| + C \\ &= \log|x^3-x^2+1| + C\end{aligned}$$

(1-7) $\displaystyle\int \cos x(\sin^2 x + \sin x + 1)\,\mathrm{d}x$

省略したやり方：

$$\int \cos x(\sin^2 x + \sin x + 1)\,\mathrm{d}x = \int (\sin x)'(\sin^2 x + \sin x + 1)\,\mathrm{d}x$$

$$= \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + \sin x + C$$

置換積分をきちんと書くと：

$t = \sin x$ とおく．$\dfrac{\mathrm{d}t}{\mathrm{d}x} = \cos x$ より，$\mathrm{d}t = \cos x \mathrm{d}x$ なので，

$$\begin{aligned}
\int \cos x(\sin^2 x + \sin x + 1)\,\mathrm{d}x &= \int (t^2 + t + 1)\mathrm{d}t \\
&= \frac{1}{3}t^3 + \frac{1}{2}t^2 + t + C \\
&= \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + \sin x + C
\end{aligned}$$

(1-8)

$$\begin{aligned}
\int \frac{(\log x)^3}{x}\,\mathrm{d}x &= \int (\log x)^3(\log x)'\,\mathrm{d}x \\
&= \frac{1}{4}(\log x)^4 + C
\end{aligned}$$

(1-9)

$$\begin{aligned}
\int \tan x\,\mathrm{d}x &= \int \frac{\sin x}{\cos x}\,\mathrm{d}x \\
&= -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x \\
&= -\int \frac{(\cos x)'}{\cos x}\,\mathrm{d}x \\
&= -\log|\cos x| + C
\end{aligned}$$

(1-10)

$$\begin{aligned}
\int \cos x e^{\sin x}\,\mathrm{d}x &= \int (\sin x)' e^{\sin x}\,\mathrm{d}x \\
&= e^{\sin x} + C
\end{aligned}$$

(1-11)

$$\begin{aligned}
\int x\sin x\,\mathrm{d}x &= \int x(-\cos x)'\,\mathrm{d}x \\
&= x(-\cos x) - \int (x)'(-\cos x)\,\mathrm{d}x \\
&= -x\cos x + \int \cos x\mathrm{d}x \\
&= -x\cos x + \sin x + C
\end{aligned}$$

(1-12)

$$\begin{aligned}
\int x\log x\,\mathrm{d}x &= \int \left(\frac{1}{2}x^2\right)' \log x\,\mathrm{d}x \\
&= \frac{1}{2}x^2\log x - \int \frac{1}{2}x^2(\log x)'\,\mathrm{d}x \\
&= \frac{1}{2}x^2\log x - \int \frac{1}{2}x^2\frac{1}{x}\,\mathrm{d}x \\
&= \frac{1}{2}x^2\log x - \int \frac{1}{2}x\,\mathrm{d}x \\
&= \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C
\end{aligned}$$

(1-13)

$$\begin{aligned}\int \frac{1}{x\log x}\,\mathrm{d}x &= \int \frac{(\log x)'}{\log x}\,\mathrm{d}x \\ &= \log|\log x| + C\end{aligned}$$

(1-14)

$$\begin{aligned}\int \log x\,\mathrm{d}x &= \int (x)'\log x\,\mathrm{d}x \\ &= x\log x - \int x(\log x)'\,\mathrm{d}x \\ &= x\log x - \int x\frac{1}{x}\,\mathrm{d}x \\ &= x\log x - \int \mathrm{d}x \\ &= x\log x - x + C\end{aligned}$$

(1-15) $\displaystyle\int \frac{1}{x^2-1}\,\mathrm{d}x$

$\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)}$ を部分分数分解する．

$\dfrac{1}{x^2-1} = \dfrac{a}{x+1} + \dfrac{b}{x-1}$ という形において，a と b を決定する．

$$\begin{aligned}\frac{a}{x+1} + \frac{b}{x-1} &= \frac{a(x-1)+b(x+1)}{(x+1)(x-1)} \\ &= \frac{(a+b)x+(-a+b)}{x^2-1}\end{aligned}$$

となるので，恒等式 $1 = (a+b)x + (-a+b)$ において係数を比較して，

$$\begin{aligned}a+b &= 0 \\ -a+b &= 1\end{aligned}$$

を得る．これを解いて，$a=-1/2,\ b=1/2$ となるので結局，

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

がわかる．

$$\begin{aligned}\int \frac{1}{x^2-1}\,\mathrm{d}x &= \frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\mathrm{d}x \\ &= \frac{1}{2}\left(\log|x-1| - \log|x+1|\right) + C \\ &= \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C\end{aligned}$$

(1-16) $\displaystyle\int e^x \sin x\,\mathrm{d}x$

部分積分による方法：

$I = \displaystyle\int e^x \sin x\,\mathrm{d}x$ とおく．

$$I = \int (e^x)'\sin x\,\mathrm{d}x$$

$$
\begin{aligned}
&= e^x \sin x - \int e^x (\sin x)' \mathrm{d}x \\
&= e^x \sin x - \int e^x \cos x \mathrm{d}x \\
&= e^x \sin x - \int (e^x)' \cos x \mathrm{d}x \\
&= e^x \sin x - e^x \cos x + \int e^x (\cos x)' \mathrm{d}x \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x \mathrm{d}x \\
&= e^x \sin x - e^x \cos x - I
\end{aligned}
$$

より，$I = e^x \sin x - e^x \cos x - I$ を I について解いて，

$$
I = \frac{1}{2} e^x (\sin x - \cos x)
$$

を得る．
オイラーの公式による方法：

$$
\begin{aligned}
\int e^x \cos x \, \mathrm{d}x + \mathrm{i} \int e^x \sin x \, \mathrm{d}x &= \int e^x (\cos x + \mathrm{i} \sin x) \, \mathrm{d}x \\
&= \int e^x e^{\mathrm{i}x} \, \mathrm{d}x \\
&= \int e^{(1+\mathrm{i})x} \, \mathrm{d}x \\
&= \frac{1}{1+\mathrm{i}} e^{(1+\mathrm{i})x} \\
&= \frac{1-\mathrm{i}}{2} e^x (\cos x + \mathrm{i} \sin x) \\
&= \frac{1}{2} e^x \left\{ (\sin x + \cos x) + \mathrm{i}(\sin x - \cos x) \right\}
\end{aligned}
$$

より，実部と虚部を比較して，

$$
\begin{aligned}
\int e^x \cos x \, \mathrm{d}x &= \frac{1}{2} e^x (\sin x + \cos x) \\
\int e^x \sin x \, \mathrm{d}x &= \frac{1}{2} e^x (\sin x - \cos x)
\end{aligned}
$$

を得る．

問題 2 以下の定積分を計算せよ．

(2-1)

$$
\begin{aligned}
\int_\alpha^\beta (x-\alpha)(x-\beta) \mathrm{d}x = \int_\alpha^\beta \{x^2 - (\alpha+\beta)x + \alpha\beta\} \mathrm{d}x &= \left[\frac{1}{3} x^3 - \frac{1}{2}(\alpha+\beta)x^2 + \alpha\beta x \right]_\alpha^\beta \\
&= \left(\frac{1}{3}\beta^3 - \frac{1}{2}(\beta+\alpha)\beta^2 + \beta^2\alpha \right) - \left(\frac{1}{3}\alpha^3 - \frac{1}{2}(\beta+\alpha)\alpha^2 + \beta\alpha^2 \right) \\
&= \frac{1}{3}(\beta^3 - \alpha^3) - \frac{1}{2}(\beta+\alpha)(\beta^2 - \alpha^2) + \beta\alpha(\beta - \alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{3}(\beta-\alpha)(\beta^2+\beta\alpha+\alpha^2) - \frac{1}{2}(\beta+\alpha)^2(\beta-\alpha) + \beta\alpha(\beta-\alpha) \\
&= \frac{1}{6}(\beta-\alpha)\{2(\beta^2+\beta\alpha+\alpha^2) - 3(\beta+\alpha)^2 + 6\beta\alpha\} \\
&= \frac{1}{6}(\beta-\alpha)(-\beta^2+2\beta\alpha-\alpha^2) \\
&= -\frac{1}{6}(\beta-\alpha)^3
\end{aligned}
$$

(2-2)

$$\int_0^\pi \sin x \mathrm{d}x = [-\cos x]_0^\pi = -\cos\pi + \cos 0 = 2$$

(2-3)

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \left[-e^{-x}\right]_0^\infty .$$

ここで,

$$\lim_{x\to\infty} e^{-x} = 0$$

より,

$$\int_0^\infty e^{-x}\,\mathrm{d}x = 1.$$

(2-4)

$$
\begin{aligned}
\int_0^\infty x e^{-x^2}\,\mathrm{d}x &= \int_0^\infty \left(-\frac{1}{2}\right)(-2x)e^{-x^2}\,\mathrm{d}x \\
&= \int_0^\infty \left(-\frac{1}{2}\right)(-x^2)' e^{-x^2}\,\mathrm{d}x \\
&= \left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^\infty \\
&= \left(-\frac{1}{2}\right)(0-1) \\
&= \frac{1}{2}
\end{aligned}
$$

(2-5)

$$
\begin{aligned}
\int_0^\infty x e^{-x}\,\mathrm{d}x &= \int_0^\infty x(-e^{-x})'\,\mathrm{d}x \\
&= \left[x(-e^{-x})\right]_0^\infty - \int_0^\infty (x)'(-e^{-x})\,\mathrm{d}x \\
&= -\left[x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\,\mathrm{d}x \\
&= -\left[x e^{-x}\right]_0^\infty - \left[e^{-x}\right]_0^\infty
\end{aligned}
$$

ここで,

$$
\begin{aligned}
&\lim_{x\to\infty} e^{-x} = 0, \\
&\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{(x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0, \\
&\lim_{x\to 0} x e^{-x} = 0 \times 1 = 0
\end{aligned}
$$

より，

$$\begin{aligned}\int_0^\infty xe^{-x}\,\mathrm{d}x &= -\left[xe^{-x}\right]_0^\infty - \left[e^{-x}\right]_0^\infty \\ &= (0-0)-(0-1)=1\end{aligned}$$

(2-6)

$$\begin{aligned}I &= \int_0^\infty e^{-x}\sin x\,\mathrm{d}x = \int_0^\infty (-e^{-x})'\sin x\,\mathrm{d}x = \left[-e^{-x}\sin x\right]_0^\infty + \int_0^\infty e^{-x}\cos x\,\mathrm{d}x \\ &= \int_0^\infty e^{-x}\cos x\,\mathrm{d}x = \int_0^\infty (-e^{-x})'\cos x\,\mathrm{d}x \\ &= \left[-e^{-x}\cos x\right]_0^\infty - \int_0^\infty e^{-x}\sin x\,\mathrm{d}x = 1 - I\end{aligned}$$

より，$I = 1 - I$ なので $I = \dfrac{1}{2}$ を得る．

問題 3 関数 $f(x)$ が

$$f(x) = x - \int_0^1 f(x)\mathrm{d}x$$

を満たすとき，$f(x)$ を求めよ．

定積分 $\displaystyle\int_0^1 f(x)\mathrm{d}x$ は定数なので，$a = \displaystyle\int_0^1 f(x)\mathrm{d}x$ とおくと，$f(x) = x - a$ なので，

$$a = \int_0^1 f(x)\mathrm{d}x = \int_0^1 (x-a)\mathrm{d}x = \left[\frac{1}{2}x^2 - ax\right]_0^1 = \frac{1}{2} - a$$

より，$a = \dfrac{1}{4}$ を得る．

問題 4 2 つの不定積分

$$A = \int e^{ax}\cos bx\mathrm{d}x, \quad B = \int e^{ax}\sin bx\mathrm{d}x$$

を求めたい．ただし，a, b は定数とする．

(4-1) 部分積分することにより，A を B の式で，B を A の式で表せ．

$$\begin{aligned}A &= \int e^{ax}\cos bx\mathrm{d}x = \frac{1}{a}\int (e^{ax})'\cos bx\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}B, \\ B &= \int e^{ax}\sin bx\mathrm{d}x = \frac{1}{a}\int (e^{ax})'\sin bx\mathrm{d}x = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}A.\end{aligned}$$

(4-2) 上で求めた A, B に対する連立方程式を解くことにより，A, B を求めよ．

$$\begin{aligned}&A = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}B = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}A\right), \\ &a^2A = ae^{ax}\cos bx + b\left(e^{ax}\sin bx - bA\right), \\ &A = \frac{1}{a^2+b^2}e^{ax}\left(a\cos bx + b\sin bx\right).\end{aligned}$$

同様に，

$$B = \frac{1}{a^2+b^2} e^{ax} \left(-b\cos bx + a\sin bx \right).$$

(4-3) 積分 $\displaystyle\int e^{(a+\mathrm{i}b)x}\mathrm{d}x$ を計算し，実部と虚部をとることにより A, B を求めよ．

$$\begin{aligned}\int e^{(a+\mathrm{i}b)x}\mathrm{d}x &= \frac{1}{a+\mathrm{i}b} e^{(a+\mathrm{i}b)x} = \frac{1}{a+\mathrm{i}b} e^{ax} e^{\mathrm{i}bx} = \frac{1}{a+\mathrm{i}b} e^{ax} (\cos bx + \mathrm{i}\sin bx) \\ &= \frac{1}{a^2+b^2} e^{ax} (a-\mathrm{i}b)(\cos bx + \mathrm{i}\sin bx) \\ &= \frac{1}{a^2+b^2} e^{ax} \{(a\cos bx + b\sin bx) + \mathrm{i}(-b\cos bx + a\sin bx)\}.\end{aligned}$$

ここで，$\displaystyle\int e^{(a+\mathrm{i}b)x}\mathrm{d}x = A + \mathrm{i}B$ なので，実部と虚部を比べて上と同じ結果を得る．