

問題1 以下で与えられる x の関数をマクローリン展開し，最初の第4項までを具体的に書き下せ．

マクローリン展開：

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \dots$$

(1-1) $f(x) = e^x, f(0) = 1,$
 $f'(x) = e^x, f'(0) = 1,$
 $f''(x) = e^x, f''(0) = 1,$
 $f'''(x) = e^x, f'''(0) = 1,$

$$f^{(4)}(x) = -(3!)(1+x)^{-4}, f^{(4)}(0) = -3!,$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots$$

(1-5) $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}, f(0) = 1,$
 $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}, f'(0) = \frac{1}{2},$

(1-2) $f(x) = \sin x, f(0) = 0,$
 $f'(x) = \cos x, f'(0) = 1,$
 $f''(x) = -\sin x, f''(0) = 0,$
 $f'''(x) = -\cos x, f'''(0) = -1,$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}, f''(0) = -\frac{1}{4},$$

$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}, f'''(0) = \frac{3}{8},$$

以下この4つの繰り返し

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

(1-3) $f(x) = \cos x, f(0) = 1,$
 $f'(x) = -\sin x, f'(0) = 0,$
 $f''(x) = -\cos x, f''(0) = -1,$
 $f'''(x) = \sin x, f'''(0) = 0,$

(1-6) $f(x) = \frac{1}{1-x} = (1-x)^{-1}, f(0) = 1,$
 $f'(x) = (-1)(1-x)^{-2}(1-x)' = (1-x)^{-2},$
 $f'(0) = 1,$
 $f''(x) = -2(1-x)^{-3}(1-x)' = 2(1-x)^{-3},$
 $f''(0) = 2,$

以下この4つの繰り返し

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$f'''(x) = (-3)2(1-x)^{-4}(1-x)'$$

$$= (3!)(1-x)^{-4},$$

$$f'''(0) = 3!,$$

(1-4) $f(x) = \log(1+x), f(0) = 0,$
 $f'(x) = (1+x)^{-1}, f'(0) = 1,$
 $f''(x) = -(1+x)^{-2}, f''(0) = -1,$
 $f'''(x) = 2(1+x)^{-3}, f'''(0) = 2,$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

問題2 e^x の展開式に $x = i\theta$ を代入することにより，オイラーの公式 $e^{i\theta} = \cos \theta + i \sin \theta$ を示せ．

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

に $x = i\theta$ を代入して，

$$e^x = 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \dots$$

$$= 1 + i\theta - \frac{1}{2}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 \dots$$

$$= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\right)$$

$$= \cos \theta + i \sin \theta$$

を得る .

問題 3 以下の数値を与えられた精度まで計算せよ .

(3-1) $\sqrt{101}$ (小数第 5 位まで)

$$\sqrt{101} = 10\sqrt{1+0.01} \text{ なので,}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots \text{ に } x = 0.01 \text{ を代入して,}$$

$$\sqrt{101} = 10 + 5 \times 0.01 - 1.25 \times 0.01^2 + 6.25 \times 10^{-7} + \dots$$

$$= 10 + 0.05 - 0.000125$$

$$= \boxed{10.04987\dots}$$

(3-2) $\sin 0.1$ (小数第 5 位まで)

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \text{ に } x = 0.1 \text{ を代入して,}$$

$$\sin x = 0.1 - 0.00016666\dots + \dots$$

$$= \boxed{0.09983\dots}$$

(3-3) e (小数第 2 位まで)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots \text{ に } x = 1 \text{ を代入して,}$$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

$$= 1 + 1 + 0.5 + 0.16666 + 0.0416666 + 0.00833333 + \dots$$

$$= \boxed{2.71\dots}$$

(3-4) \sqrt{e} (小数第 2 位まで)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots \text{ に } x = 1/2 \text{ を代入して,}$$

$$e = 1 + 0.5 + 0.125 + 0.02083 + \dots$$

$$= \boxed{1.64\dots}$$

(3-5) $\tan^{-1} \frac{1}{5}$ (小数第 3 位まで)

$$f(x) = \tan^{-1} x, f(0) = 0,$$

$$f'(x) = (1+x^2)^{-1}, f'(0) = 1,$$

$$f''(x) = -2x(1+x^2)^{-2}, f''(0) = 0,$$

$$f'''(x) = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}, f'''(0) = -2,$$

$$\text{よ} \text{り, } \tan^{-1} x = x - \frac{1}{3}x^3 + \dots$$

$$\text{この展開式に } x = 1/5 \text{ を代入して,}$$

$$\tan^{-1} \frac{1}{5} = \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \dots = 0.2 - 0.002666\dots + \dots = \boxed{0.197\dots}$$

(3-6) $\log 2$ (小数第 3 位まで)

$$\left(\log \frac{1+x}{1-x} \text{ の展開式を用いよ} \right)$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \text{ より,}$$

$\log(1-x) = -\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots\right)$ なので,

$\log \frac{1+x}{1-x} = \log(1+x) - \log(1-x) = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots\right)$ と展開される.

ここで, $\frac{1+x}{1-x} = 2$ を解くと $x = 1/3$ なので,

$$\log 2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \dots\right) = \frac{2}{3} + \frac{2}{3^4} + \frac{2}{5 \cdot 3^5} + \dots = \boxed{0.693 \dots}$$

問題 4 $\sqrt[3]{2}$ の値をできるだけ正確に知りたい.

(4-1) $\sqrt[3]{2} = 1.25 \times (1.024)^{\frac{1}{3}}$ を示せ.

$$1.25 \times (1.024)^{\frac{1}{3}} = \frac{5}{4} \left(\frac{1024}{1000}\right)^{\frac{1}{3}} = \frac{5}{4} \left(\frac{2^{10}}{10^3}\right)^{\frac{1}{3}} = \frac{5}{4} \frac{2^{\frac{10}{3}}}{10} = \frac{2^{\frac{10}{3}}}{2^3} = 2^{\frac{10}{3}-3} = 2^{\frac{1}{3}} = \sqrt[3]{2}$$

(4-2) $(1+x)^{\frac{1}{3}}$ を x の 2 次までマクローリン展開せよ.

$$\begin{aligned} (1+x)^{\frac{1}{3}} &= 1 + \frac{1}{3}x + \frac{1}{2} \cdot \frac{1}{3} \left(\frac{1}{3} - 1\right) x^2 + \dots \\ &= \boxed{1 + \frac{1}{3}x - \frac{1}{9}x^2 + \dots} \end{aligned}$$

(4-3) $\sqrt[3]{2}$ の値を小数第 5 位まで求めよ.

$$\begin{aligned} \sqrt[3]{2} &= 1.25 \times (1.024)^{\frac{1}{3}} = 1.25 \times \left(1 + \frac{1}{3}0.024 - \frac{1}{9}0.024^2 + \dots\right) \\ &= 1.25 \times (1 + 0.008 - 0.0008^2 + \dots) \\ &= 1.25 + 0.01 - 0.00008 + \dots \\ &= \boxed{1.25992 \dots} \end{aligned}$$

問題 5 以下の極限值を計算せよ.

(5-1) $\lim_{x \rightarrow \infty} e^x = \boxed{\infty}$

(5-2) $\lim_{x \rightarrow \infty} e^{-x} = \boxed{0}$

(5-3) $\lim_{x \rightarrow \infty} \log x = \boxed{\infty}$

(5-4) $\lim_{x \rightarrow \infty} \frac{\log x}{x} = \lim_{x \rightarrow \infty} \frac{(\log x)'}{(x)'} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = \boxed{0}$

(5-5) $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \lim_{x \rightarrow 0} \frac{\{\log(1+x)\}'}{(x)'} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{1} = \lim_{x \rightarrow 0} \frac{1}{1+x} = \boxed{1}$

(5-6) $\lim_{x \rightarrow 0} \frac{\log(1+x) - x}{x^2} = \lim_{x \rightarrow 0} \frac{\{\log(1+x) - x\}'}{(x^2)'} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - 1}{2x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - \frac{1+x}{1+x}}{2x}$
 $= \lim_{x \rightarrow 0} \frac{\frac{1-(1+x)}{1+x}}{2x} = \lim_{x \rightarrow 0} \frac{\frac{-x}{1+x}}{2x} = \lim_{x \rightarrow 0} \frac{-1}{2(1+x)} = \boxed{-\frac{1}{2}}$

(5-7) $\lim_{x \rightarrow 0} x e^{-x} = \left(\lim_{x \rightarrow 0} x\right) \left(\lim_{x \rightarrow 0} e^{-x}\right) = 0 \times 1 = \boxed{0}$

$$(5-8) \quad \lim_{x \rightarrow \infty} x e^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{(x)'}{(e^x)'} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = \boxed{0}$$

$$(5-9) \quad \lim_{x \rightarrow 0} \frac{x}{e^x + 1} = \frac{0}{1 + 1} = \boxed{0}$$

$$(5-10) \quad \lim_{x \rightarrow 0} \frac{x}{e^x - 1} = \lim_{x \rightarrow 0} \frac{(x)'}{(e^x - 1)'} = \lim_{x \rightarrow 0} \frac{1}{e^x} = \boxed{1}$$

$$(5-11) \quad \lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{(\sin x - x)'}{(x^3)'} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{(\cos x - 1)'}{(3x^2)'} \\ = \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = \lim_{x \rightarrow 0} \frac{(-\sin x)'}{(6x)'} = \lim_{x \rightarrow 0} \frac{-\cos x}{6} = \boxed{-\frac{1}{6}}$$

$$(5-12) \quad \lim_{x \rightarrow \infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right) = \lim_{x \rightarrow \infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{(\frac{\pi}{2} - \tan^{-1} x)'}{(\frac{1}{x})'} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\ = \lim_{x \rightarrow \infty} \frac{x^2}{1+x^2} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x^2} + 1} = \frac{1}{0 + 1} = \boxed{1}$$

$$(5-13) \quad \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x + x^2} \right) = \lim_{x \rightarrow 0} \frac{x + x^2 - \sin x}{(x + x^2) \sin x} = \lim_{x \rightarrow 0} \frac{(x + x^2 - \sin x)'}{\{(x + x^2) \sin x\}'} \\ = \lim_{x \rightarrow 0} \frac{1 + 2x - \cos x}{1 + 2x - \cos x} = \lim_{x \rightarrow 0} \frac{2 + \sin x}{2 \sin x + 2(1 + 2x) \cos x - (x + x^2) \sin x} = \boxed{1}$$