数学演習 I　第 12 回　微分法 II

2015 年 7 月 1 日　担当：佐藤 純

問題 1

(1-1) 積の微分公式 $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$ を示せ。

$$
\begin{aligned}
\{f(x)g(x)\}' &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{\{f(x+h) - f(x)\}g(x+h) + f(x)\{g(x+h) - g(x)\}}{h} \\
&= \left(\lim_{h\to 0}\frac{\{f(x+h) - f(x)\}}{h}\right)\left(\lim_{h\to 0} g(x+h)\right) + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

(1-2) 商の微分公式 $\left\{\dfrac{f(x)}{g(x)}\right\}' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ を示せ。

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\frac{1}{g} + f\left(\frac{1}{g}\right)'$$

ここで、

$$\left(\frac{1}{g}\right)' = \left(g^{-1}\right)' = (-1)g^{-2}\cdot g' = -\frac{g'}{g^2}$$

なので、

$$\left(\frac{f}{g}\right)' = f'\frac{1}{g} + f\left(-\frac{g'}{g^2}\right) = \frac{f'g - fg'}{g^2}$$

を得る。

(1-3) 対数微分の公式 $f'(x) = f(x)\{\log f(x)\}'$ を示せ。

$$\text{右辺} = f(\log f)' = f\frac{1}{f}f' = f' = \text{左辺}.$$

問題 2 以下で与えられた関数 y を x で微分し，y' を求めよ．ただし，a, b は定数とする．

(2-1) $y = (x^2+1)^5(3x+1)^4$

$$
\begin{aligned}
\left\{(x^2+1)^5(3x+1)^4\right\}' &= \left\{(x^2+1)^5\right\}'(3x+1)^4 + (x^2+1)^5\left\{(3x+1)^4\right\}' \\
&= 5(x^2+1)^4(x^2+1)'(3x+1)^4 + (x^2+1)^5 4(3x+1)^3(3x+1)' \\
&= \boxed{10x(x^2+1)^4(3x+1)^4 + 12(x^2+1)^5(3x+1)^3}
\end{aligned}
$$

(2-2) $(e^x)' = \boxed{e^x}$

(2-3) $(\log x)' = \boxed{\dfrac{1}{x}}$

(2-4) $(\sin x)' = \boxed{\cos x}$

(2-5) $(\cos x)' = \boxed{-\sin x}$

(2-6) $y = x + \sqrt{x^2 + a}$

$$\begin{aligned}\left(x + \sqrt{x^2+a}\right)' &= (x)' + \left(\sqrt{x^2+a}\right)' \\ &= 1 + \left\{(x^2+a)^{\frac{1}{2}}\right\}' \\ &= 1 + \frac{1}{2}(x^2+a)^{-\frac{1}{2}}(x^2+a)' \\ &= \boxed{1 + \frac{x}{\sqrt{x^2+a}}}\end{aligned}$$

(2-7) $y = e^{ax}$

$$(e^{ax})' = e^{ax}(ax)' = \boxed{ae^{ax}}$$

(2-8) $y = e^{-x^2}$

$$\left(e^{-x^2}\right)' = e^{-x^2}(-x^2)' = \boxed{-2xe^{-x^2}}$$

(2-9) $y = \tan x$

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \\ &= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \boxed{\frac{1}{\cos^2 x}} \qquad (1) \\ &= \boxed{1 + \tan^2 x} \qquad (2)\end{aligned}$$

今後、(1), (2) 式の両方を使う。

(2-10) $y = \sin^{-1} x$

$$\begin{aligned}&x = \sin y, \\ &\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}, \\ &(\sin^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \boxed{\frac{1}{\sqrt{1-x^2}}}\end{aligned}$$

(2-11) $y = \cos^{-1} x$

$$\begin{aligned}&x = \cos y, \\ &\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}, \\ &(\cos^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \boxed{-\frac{1}{\sqrt{1-x^2}}}\end{aligned}$$

(2-12) $y = \tan^{-1} x$

$$\begin{aligned} &x = \tan y, \\ &\frac{\mathrm{d}x}{\mathrm{d}y} = 1 + \tan^2 y = 1 + x^2, \\ &(\tan^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \boxed{\frac{1}{1+x^2}} \end{aligned}$$

(2-13) $y = 3^x$

$$(3^x)' = y(\log y)' = 3^x(x \log 3)' = \boxed{3^x \log 3}$$

(2-14) $y = 2^x + \cos x$

$$(2^x + \cos x)' = 2^x(x \log 2)' - \sin x = \boxed{2^x \log 2 - \sin x}$$

(2-15) $y = \pi \sin x + \cos x$

$$(\pi \sin x + \cos x)' = \pi(\sin x)' + (\cos x)' = \boxed{\pi \cos x - \sin x}$$

(2-16) $y = e \log x - e$

$$y' = \boxed{\frac{e}{x}}$$

(2-17) $y = \log(2x)$

$$y' = (\log x + \log 2)' = \boxed{\frac{1}{x}}$$

(2-18) $y = \log_{10} x$

$$y' = \left(\frac{\log x}{\log 10}\right)' = \boxed{\frac{1}{x \log 10}}$$

(2-19) $y = \log x - \log \pi$

$$y' = \boxed{\frac{1}{x}}$$

(2-20) $y = x^2 \log 2$

$$y' = \boxed{2x \log 2}$$

(2-21) $y = x^2 \log x$

$$y' = (x^2)' \log x + x^2(\log x)' = \boxed{2x \log x + x}$$

(2-22) $y = \dfrac{\sin x}{e^x}$

$$y' = (e^{-x} \sin x)' = (e^{-x})' \sin x + e^{-x}(\sin x)' = -e^{-x} \sin x + e^{-x} \cos x = \boxed{e^{-x}(\cos x - \sin x)}$$

(2-23) $y = \dfrac{x+1}{x-1}$

$$y' = \left(1 + \frac{2}{x-1}\right)' = \left(\frac{2}{x-1}\right)' = \boxed{-\frac{2}{(x-1)^2}}$$

(2-24) $y = \dfrac{\log x}{x}$

$$y' = \frac{(\log x)'x - \log x(x)'}{x^2} = \boxed{\frac{1-\log x}{x^2}}$$

(2-25) $y = (1+x^2)e^{-x}$

$$\begin{aligned}\left\{(1+x^2)e^{-x}\right\}' &= \left\{(1+x^2)\right\}' e^{-x} + (1+x^2)\left(e^{-x}\right)' \\ &= 2xe^{-x} + (1+x^2)e^{-x}(-1) \\ &= (2x-1-x^2)e^{-x} \\ &= \boxed{-(x-1)^2e^{-x}}\end{aligned}$$

(2-26) $y = e^{\sin x}$

$$\left(e^{\sin x}\right)' = e^{\sin x}(\sin x)' = \boxed{\cos x e^{\sin x}}$$

(2-27) $y = x^x$

$$(x^x)' = y(\log y)' = x^x(x\log x)' = x^x\left\{(x)'\log x + x(\log x)'\right\} = \boxed{(1+\log x)x^x}$$

(2-28) $y = x^{\sin x}$

$$\begin{aligned}&(x^{\sin x})' = y(\log y)' = x^{\sin x}(\sin x\log x)' = x^{\sin x}\left\{(\sin x)'\log x + \sin x(\log x)'\right\} \\ &= \boxed{\left(\cos x\log x + \frac{\sin x}{x}\right)x^{\sin x}}\end{aligned}$$

(2-29) $y = e^{ax}\cos bx$

$$\begin{aligned}(e^{ax}\cos bx)' &= (e^{ax})'\cos bx + e^{ax}(\cos bx)' \\ &= \boxed{e^{ax}(a\cos bx - b\sin bx)}\end{aligned}$$

(2-30) $y = (\log x)^3$

$$\begin{aligned}\left\{(\log x)^3\right\}' &= 3(\log x)^2(\log x)' \\ &= \boxed{\frac{3(\log x)^2}{x}}\end{aligned}$$

(2-31) $y = \dfrac{1}{\sqrt{\log x}}$

$$\begin{aligned}\left(\frac{1}{\sqrt{\log x}}\right)' &= \left\{(\log x)^{-\frac{1}{2}}\right\}' \\ &= -\frac{1}{2}(\log x)^{-\frac{3}{2}}(\log x)' \\ &= \boxed{-\frac{1}{2x\log x\sqrt{\log x}}}\end{aligned}$$

(2-32) $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

より，

$$1 - \cos 2x = 2\sin^2 x, \quad 1 + \cos 2x = 2\cos^2 x$$

なので，

$$\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)' = (\tan x)' = \boxed{\frac{1}{\cos^2 x}}$$

(2-33) $y = e^{ax}\dfrac{a\cos bx + b\sin bx}{a^2+b^2}$

$$\begin{aligned}\left(e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)' &= (e^{ax})'\frac{a\cos bx + b\sin bx}{a^2+b^2} + e^{ax}\left(\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)' \\ &= e^{ax}\frac{a^2\cos bx + ab\sin bx}{a^2+b^2} + e^{ax}\frac{-ab\sin bx + b^2\cos bx}{a^2+b^2} \\ &= \boxed{e^{ax}\cos bx}\end{aligned}$$

(2-34) $y = x\log x - x$

$$(x\log x - x)' = (x\log x)' - (x)' = (x)'\log x + x(\log x)' - 1 = \log x + x\frac{1}{x} - 1 = \boxed{\log x}$$

(2-35) $y = x\cos^2 x$

$$\left(x\cos^2 x\right)' = \cos^2 x + x \cdot 2\cos x(-\sin x) = \boxed{\cos^2 x - x\sin 2x}$$

(2-36) $y = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right|$

$$\left(\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right)' = \frac{1}{2a}\left(\log|x-a| - \log|x+a|\right)' = \frac{1}{2a}\left(\frac{(x-a)'}{x-a} - \frac{(x+a)'}{x+a}\right) = \boxed{\frac{1}{x^2-a^2}}$$

(2-37) $y = \log\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$\begin{aligned}y' &= \{\log(\sqrt{1+x}-\sqrt{1-x}) - \log(\sqrt{1+x}+\sqrt{1-x})\}' \\ &= \frac{\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}}{\sqrt{1+x}-\sqrt{1-x}} - \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}}}{\sqrt{1+x}+\sqrt{1-x}} \\ &= \frac{1}{2}\frac{(\sqrt{1+x}+\sqrt{1-x})(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1-x}}) - (\sqrt{1+x}-\sqrt{1-x})(\frac{1}{\sqrt{1+x}}-\frac{1}{\sqrt{1-x}})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})} \\ &= \frac{\frac{\sqrt{1+x}}{\sqrt{1-x}} + \frac{\sqrt{1-x}}{\sqrt{1+x}}}{(1+x)-(1-x)} = \frac{\frac{(1+x)+(1-x)}{\sqrt{1-x^2}}}{2x} \\ &= \boxed{\frac{1}{x\sqrt{1-x^2}}}\end{aligned}$$

(2-38) $y = \log\left|x + \sqrt{x^2+1}\right|$

$$y' = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \boxed{\frac{1}{\sqrt{x^2+1}}}$$

(2-39) $y = \dfrac{1}{2}\log|\cos x + \sin x| + \dfrac{x}{2}$

$$y' = \frac{1}{2}\frac{1 - \sin x + \cos x}{\cos x + \sin x} + \frac{1}{2} = \frac{1}{2}\frac{2\cos x}{\cos x + \sin x} = \boxed{\frac{1}{1+\tan x}}$$

(2-40) $y = \dfrac{1}{\sqrt{3}}\log\left|\dfrac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|$

$$\begin{aligned}
y' &= \frac{1}{\sqrt{3}}\left[\left(\log\left|\sqrt{3}+\tan\frac{x}{2}\right|\right)' - \left(\log\left|\sqrt{3}-\tan\frac{x}{2}\right|\right)'\right] \\
&= \frac{1}{\sqrt{3}}\left[\frac{\frac{1}{\cos^2\frac{x}{2}}\frac{1}{2}}{\sqrt{3}+\tan\frac{x}{2}} - \frac{-\frac{1}{\cos^2\frac{x}{2}}\frac{1}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right] \\
&= \frac{1}{2\sqrt{3}\cos^2\frac{x}{2}}\left[\frac{1}{\sqrt{3}+\tan\frac{x}{2}} + \frac{1}{\sqrt{3}-\tan\frac{x}{2}}\right] \\
&= \frac{1}{2\sqrt{3}\cos^2\frac{x}{2}}\frac{2\sqrt{3}}{3-\tan^2\frac{x}{2}} \\
&= \frac{1}{3\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \\
&= \frac{2}{3(1+\cos x) - (1-\cos x)} \\
&= \boxed{\frac{1}{1+2\cos x}}
\end{aligned}$$

(2-41) $y = \dfrac{1}{4\sqrt{2}}\log\left|\dfrac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|$
$+\dfrac{1}{2\sqrt{2}}\left(\tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1)\right)$

$$\begin{aligned}
y' &= \frac{1}{4\sqrt{2}}\left(\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}}{1+(\sqrt{2}x+1)^2} + \frac{\sqrt{2}}{1+(\sqrt{2}x-1)^2}\right) \\
&= \frac{1}{4}\left(\frac{\sqrt{2}x+1}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{4}\left(\frac{1}{x^2+\sqrt{2}x+1} + \frac{1}{x^2-\sqrt{2}x+1}\right) \\
&= \frac{1}{4}\left(\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}\right) \\
&= \frac{1}{4}\frac{(\sqrt{2}x+2)(x^2-\sqrt{2}x+1) - (\sqrt{2}x-2)(x^2+\sqrt{2}x+1)}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} \\
&= \boxed{\frac{1}{1+x^4}}
\end{aligned}$$