数学I　第11回　関数の微分法

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問題 1

(1-1) 積の微分公式 $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$ を示せ。

$$\begin{aligned}\{f(x)g(x)\}' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{\{f(x+h)-f(x)\}g(x+h)+f(x)\{g(x+h)-g(x)\}}{h}\\ &= \left(\lim_{h\to 0}\frac{\{f(x+h)-f(x)\}}{h}\right)\left(\lim_{h\to 0}g(x+h)\right)+f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)g(x)+f(x)g'(x)\end{aligned}$$

(1-2) 商の微分公式 $\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$ を示せ。

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\frac{1}{g}+f\left(\frac{1}{g}\right)'$$

ここで、

$$\left(\frac{1}{g}\right)' = \left(g^{-1}\right)' = (-1)g^{-2}\cdot g' = -\frac{g'}{g^2}$$

なので、

$$\left(\frac{f}{g}\right)' = f'\frac{1}{g}+f\left(-\frac{g'}{g^2}\right) = \frac{f'g-fg'}{g^2}$$

を得る。

(1-3) 対数微分の公式 $f'(x) = f(x)\{\log f(x)\}'$ を示せ。

$$\text{右辺} = f(\log f)' = f\frac{1}{f}f' = f' = \text{左辺}.$$

問題 2　以下で与えられた関数 y を x で微分し，y' を求めよ．ただし，a, b は定数とする．

(2-1) $y = (x^2+1)^5(3x+1)^4$

$$\begin{aligned}\left\{(x^2+1)^5(3x+1)^4\right\}' &= \left\{(x^2+1)^5\right\}'(3x+1)^4+(x^2+1)^5\left\{(3x+1)^4\right\}'\\ &= 5(x^2+1)^4(x^2+1)'(3x+1)^4+(x^2+1)^5 4(3x+1)^3(3x+1)'\\ &= \boxed{10x(x^2+1)^4(3x+1)^4+12(x^2+1)^5(3x+1)^3}\end{aligned}$$

(2-2) $(e^x)' = \boxed{e^x}$

(2-3) $(\log x)' = \boxed{\dfrac{1}{x}}$

(2-4) $(\sin x)' = \boxed{\cos x}$

(2-5) $(\cos x)' = \boxed{-\sin x}$

(2-6) $y = x + \sqrt{x^2 + a}$

$$
\begin{aligned}
\left(x + \sqrt{x^2 + a}\right)' &= (x)' + \left(\sqrt{x^2 + a}\right)' \\
&= 1 + \left\{(x^2 + a)^{\frac{1}{2}}\right\}' \\
&= 1 + \frac{1}{2}(x^2 + a)^{-\frac{1}{2}}(x^2 + a)' \\
&= \boxed{1 + \frac{x}{\sqrt{x^2 + a}}}
\end{aligned}
$$

(2-7) $y = e^{ax}$

$$
(e^{ax})' = e^{ax}(ax)' = \boxed{ae^{ax}}
$$

(2-8) $y = e^{-x^2}$

$$
\left(e^{-x^2}\right)' = e^{-x^2}(-x^2)' = \boxed{-2xe^{-x^2}}
$$

(2-9) $y = \tan x$

$$
\begin{aligned}
(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \\
&= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \boxed{\frac{1}{\cos^2 x}} && (1) \\
&= \boxed{1 + \tan^2 x} && (2)
\end{aligned}
$$

今後、(1), (2) 式の両方を使う。

(2-10) $y = \sin ax \cos bx$

$$
\begin{aligned}
(\sin ax \cos bx)' &= (\sin ax)' \cos bx + \sin ax \,(\cos bx)' \\
&= \cos ax \,(ax)' \cos bx + \sin ax(-\sin bx)\,(bx)' \\
&= \boxed{a \cos ax \cos bx - b \sin ax \sin bx.}
\end{aligned}
$$

(2-11) $y = \sin^2 ax$

$$
\begin{aligned}
\left(\sin^2 ax\right)' &= 2 \sin ax \,(\sin ax)' \\
&= 2 \sin ax \cos ax \,(ax)' \\
&= 2a \sin ax \cos ax \\
&= \boxed{a \sin 2ax}
\end{aligned}
$$

(2-12) $y = \sin^{-1} x$

$$\begin{aligned}
&x = \sin y, \\
&\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}, \\
&(\sin^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \boxed{\frac{1}{\sqrt{1 - x^2}}}
\end{aligned}$$

(2-13) $y = \cos^{-1} x$

$$\begin{aligned}
&x = \cos y, \\
&\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}, \\
&(\cos^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \boxed{-\frac{1}{\sqrt{1 - x^2}}}
\end{aligned}$$

(2-14) $y = \tan^{-1} x$

$$\begin{aligned}
&x = \tan y, \\
&\frac{\mathrm{d}x}{\mathrm{d}y} = 1 + \tan^2 y = 1 + x^2, \\
&(\tan^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \boxed{\frac{1}{1 + x^2}}
\end{aligned}$$

(2-15) $y = (1 + x^2)e^{-x}$

$$\begin{aligned}
\left\{(1 + x^2)e^{-x}\right\}' &= \left\{(1 + x^2)\right\}' e^{-x} + (1 + x^2) \left(e^{-x}\right)' \\
&= 2xe^{-x} + (1 + x^2)e^{-x}(-1) \\
&= (2x - 1 - x^2)e^{-x} \\
&= \boxed{-(x - 1)^2 e^{-x}}
\end{aligned}$$

(2-16) $y = e^{\sin x}$

$$\left(e^{\sin x}\right)' = e^{\sin x}(\sin x)' = \boxed{\cos x e^{\sin x}}$$

(2-17) $y = x^x$

$$(x^x)' = y(\log y)' = x^x(x \log x)' = x^x \left\{(x)' \log x + x(\log x)'\right\} = \boxed{(1 + \log x)x^x}$$

(2-18) $y = x^{\sin x}$

$$\begin{aligned}
&(x^{\sin x})' = y(\log y)' = x^{\sin x}(\sin x \log x)' = x^{\sin x} \left\{(\sin x)' \log x + \sin x(\log x)'\right\} \\
&= \boxed{\left(\cos x \log x + \frac{\sin x}{x}\right) x^{\sin x}}
\end{aligned}$$

(2-19) $y = \sqrt{\sin 2x}$

$$\begin{aligned}
\left(\sqrt{\sin 2x}\right)' &= \left\{(\sin 2x)^{\frac{1}{2}}\right\}' \\
&= \frac{1}{2}(\sin 2x)^{-\frac{1}{2}}(\sin 2x)' \\
&= \frac{1}{2}(\sin 2x)^{-\frac{1}{2}}(\cos 2x)(2x)' \\
&= \boxed{\frac{\cos 2x}{\sqrt{\sin 2x}}}
\end{aligned}$$

(2-20) $y = \dfrac{x^2-1}{2}\log(1-x) - \dfrac{x}{2} - \dfrac{x^2}{4}$

$$\begin{aligned}\left(\frac{x^2-1}{2}\log(1-x) - \frac{x}{2} - \frac{x^2}{4}\right)' &= \left(\frac{x^2-1}{2}\right)'\log(1-x) + \frac{x^2-1}{2}\{\log(1-x)\}' - \frac{1}{2} - \frac{x}{2} \\ &= x\log(1-x) + \frac{x^2-1}{2}\frac{(1-x)'}{1-x} - \frac{1}{2} - \frac{x}{2} \\ &= x\log(1-x) + \frac{(x+1)(x-1)}{2}\frac{1}{x-1} - \frac{1}{2} - \frac{x}{2} \\ &= \boxed{x\log(1-x)}\end{aligned}$$

(2-21) $y = e^{ax}\cos bx$

$$\begin{aligned}(e^{ax}\cos bx)' &= (e^{ax})'\cos bx + e^{ax}(\cos bx)' \\ &= \boxed{e^{ax}(a\cos bx - b\sin bx)}\end{aligned}$$

(2-22) $y = (\log x)^3$

$$\begin{aligned}\left\{(\log x)^3\right\}' &= 3(\log x)^2(\log x)' \\ &= \boxed{\frac{3(\log x)^2}{x}}\end{aligned}$$

(2-23) $y = \dfrac{1}{\sqrt{\log x}}$

$$\begin{aligned}\left(\frac{1}{\sqrt{\log x}}\right)' &= \left\{(\log x)^{-\frac{1}{2}}\right\}' \\ &= -\frac{1}{2}(\log x)^{-\frac{3}{2}}(\log x)' \\ &= \boxed{-\frac{1}{2x\log x\sqrt{\log x}}}\end{aligned}$$

(2-24) $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

より,

$$1-\cos 2x = 2\sin^2 x, \quad 1+\cos 2x = 2\cos^2 x$$

なので,

$$\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)' = (\tan x)' = \boxed{\frac{1}{\cos^2 x}}$$

(2-25) $y = e^{ax}\dfrac{a\cos bx + b\sin bx}{a^2+b^2}$

$$\begin{aligned}\left(e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)' &= (e^{ax})'\frac{a\cos bx + b\sin bx}{a^2+b^2} + e^{ax}\left(\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)' \\ &= e^{ax}\frac{a^2\cos bx + ab\sin bx}{a^2+b^2} + e^{ax}\frac{-ab\sin bx + b^2\cos bx}{a^2+b^2} \\ &= \boxed{e^{ax}\cos bx}\end{aligned}$$

(2-26) $y = \sin x^3$

$$(\sin x^3)' = \cos x^3 (x^3)' = \boxed{3x^2 \cos x^3}$$

(2-27) $y = e^{\sin^2 ax}$

$$(e^{\sin^2 ax})' = e^{\sin^2 ax}(\sin^2 ax)' = \boxed{a \sin 2ax e^{\sin^2 ax}} \qquad \text{(問題 (2-11) を参照)}$$

(2-28) $y = x \log x - x$

$$(x \log x - x)' = (x \log x)' - (x)' = (x)' \log x + x(\log x)' - 1 = \log x + x\frac{1}{x} - 1 = \boxed{\log x}$$

(2-29) $y = \dfrac{\sin 2x}{\sqrt{1-x^2}}$

$$\begin{aligned}\left(\frac{\sin 2x}{\sqrt{1-x^2}}\right)' &= \frac{(\sin 2x)'}{\sqrt{1-x^2}} + (\sin 2x)\{(1-x^2)^{-1/2}\}' \\ &= \frac{2\cos 2x}{\sqrt{1-x^2}} + (\sin 2x)\left(-\frac{1}{2}\right)(1-x^2)^{-3/2}(1-x^2)' \\ &= \boxed{\frac{2\cos 2x}{\sqrt{1-x^2}} + \frac{x \sin 2x}{(1-x^2)\sqrt{1-x^2}}}\end{aligned}$$

(2-30) $y = x \cos^2 x$

$$\left(x \cos^2 x\right)' = \cos^2 x + x \cdot 2\cos x(-\sin x) = \boxed{\cos^2 x - x \sin 2x}$$

(2-31) $y = \sin^2 x^3$

$$\left(\sin^2 x^3\right)' = 2\sin x^3 (\sin x^3)' = 2 \sin x^3 \cos x^3 \cdot 3x^2 = 6x^2 \sin x^3 \cos x^3 = \boxed{3x^2 \sin 2x^3}$$

(2-32) $y = x^{\sqrt{x}}$

$$\left(x^{\sqrt{x}}\right)' = y(\log y)' = x^{\sqrt{x}}(\sqrt{x}\log x)' = x^{\sqrt{x}}\left(\frac{1}{2}\frac{1}{\sqrt{x}}\log x + \sqrt{x}\frac{1}{x}\right) = \boxed{x^{\sqrt{x}}\,\frac{\log x + 2}{2\sqrt{x}}}$$

(2-33) $y = \sqrt[3]{\dfrac{x-1}{x+1}}$

$$\left(\sqrt[3]{\frac{x-1}{x+1}}\right)' = \frac{1}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}\left(\frac{x-1}{x+1}\right)' = \frac{1}{3}\left(\frac{x-1}{x+1}\right)^{-\frac{2}{3}}\frac{2}{(x+1)^2} = \boxed{\frac{2}{3(x-1)^{\frac{2}{3}}(x+1)^{\frac{4}{3}}}}$$

(2-34) $y = \log\left|\tan\dfrac{x}{2}\right|$

$$\left(\log\left|\tan\frac{x}{2}\right|\right)' = \frac{\left(\tan\frac{x}{2}\right)'}{\tan\frac{x}{2}} = \frac{\frac{1}{\cos^2\frac{x}{2}}\left(\frac{x}{2}\right)'}{\tan\frac{x}{2}} = \frac{1}{2\cos^2\frac{x}{2}\tan\frac{x}{2}} = \frac{1}{2\cos\frac{x}{2}\sin\frac{x}{2}} = \boxed{\frac{1}{\sin x}}$$

(2-35) $y = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right|$

$$\left(\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right)' = \frac{1}{2a}\left(\log|x-a| - \log|x+a|\right)' = \frac{1}{2a}\left(\frac{(x-a)'}{x-a} - \frac{(x+a)'}{x+a}\right) = \boxed{\frac{1}{x^2-a^2}}$$

(2-36) $y = \frac{1}{2}(x\sqrt{x^2+a} + a\log|x+\sqrt{x^2+a}|)$

$$\begin{aligned} y' &= \frac{1}{2}\left(\sqrt{x^2+a} + x\frac{1}{2}\frac{2x}{\sqrt{x^2+a}} + a\frac{1+\frac{1}{2}\frac{2x}{\sqrt{x^2+a}}}{x+\sqrt{x^2+a}}\right) \\ &= \frac{1}{2}\left(\sqrt{x^2+a} + \frac{x^2}{\sqrt{x^2+a}} + a\frac{\frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}}}{x+\sqrt{x^2+a}}\right) \\ &= \frac{1}{2}\left(\sqrt{x^2+a} + \frac{x^2}{\sqrt{x^2+a}} + \frac{a}{\sqrt{x^2+a}}\right) \\ &= \boxed{\sqrt{x^2+a}} \end{aligned}$$

(2-37) $y = \log\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$\begin{aligned} y' &= \{\log(\sqrt{1+x}-\sqrt{1-x}) - \log(\sqrt{1+x}+\sqrt{1-x})\}' \\ &= \frac{\frac{1}{2\sqrt{1+x}}+\frac{1}{2\sqrt{1-x}}}{\sqrt{1+x}-\sqrt{1-x}} - \frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2\sqrt{1-x}}}{\sqrt{1+x}+\sqrt{1-x}} \\ &= \frac{1}{2}\frac{(\sqrt{1+x}+\sqrt{1-x})(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1-x}}) - (\sqrt{1+x}-\sqrt{1-x})(\frac{1}{\sqrt{1+x}}-\frac{1}{\sqrt{1-x}})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})} \\ &= \frac{\frac{\sqrt{1+x}}{\sqrt{1-x}}+\frac{\sqrt{1-x}}{\sqrt{1+x}}}{(1+x)-(1-x)} = \frac{\frac{(1+x)+(1-x)}{\sqrt{1-x^2}}}{2x} \\ &= \boxed{\frac{1}{x\sqrt{1-x^2}}} \end{aligned}$$

(2-38) $y = \log\left|x+\sqrt{x^2+1}\right|$

$$y' = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \boxed{\frac{1}{\sqrt{x^2+1}}}$$

(2-39) $y = \frac{1}{3}\tan^3 x + 3\tan x - \frac{3}{\tan x} - \frac{1}{3\tan^3 x}$

$t = \tan x,\ c = \cos x,\ s = \sin x$ と略記する．$t' = \frac{1}{c^2}$ である．

$$\begin{aligned} y' &= \left(\frac{1}{3}t^3 + 3t - \frac{3}{t} - \frac{1}{3t^3}\right)' = t^2t' + 3t' + 3\frac{1}{t^2}t' + \frac{1}{t^4}t' = \frac{1}{t^4c^2}(t^6+3t^4+3t^2+1) \\ &= \frac{1}{t^4c^2}(1+t^2)^3 = \frac{1}{t^4c^2}\left(\frac{1}{c^2}\right)^3 = \boxed{\frac{1}{\cos^4 x\sin^4 x}} \end{aligned}$$

(2-40) $y = \frac{1}{2}\log|\cos x + \sin x| + \frac{x}{2}$

$$y' = \frac{1}{2}\frac{-\sin x + \cos x}{\cos x + \sin x} + \frac{1}{2} = \frac{1}{2}\frac{2\cos x}{\cos x + \sin x} = \boxed{\frac{1}{1+\tan x}}$$

(2-41) $y = x\sin^{-1}x + \sqrt{1-x^2}$

$$y' = \sin^{-1}x + x\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x) = \boxed{\sin^{-1}x}$$

(2-42) $y = 2a\tan^{-1}\sqrt{\dfrac{a+x}{a-x}} - \sqrt{a^2-x^2}$

$$y' = 2a\frac{\frac{1}{2}\sqrt{\frac{a-x}{a+x}}\frac{2a}{(a-x)^2}}{1+\frac{a+x}{a-x}} - \frac{1}{2}\frac{-2x}{\sqrt{a^2-x^2}} = 2a\frac{\sqrt{\frac{a-x}{a+x}}\frac{a}{(a-x)^2}}{\frac{2a}{a-x}} + \frac{x}{\sqrt{a^2-x^2}}$$
$$= \sqrt{\frac{a-x}{a+x}}\frac{a}{a-x} + \frac{x}{\sqrt{a^2-x^2}} = \boxed{\sqrt{\frac{a+x}{a-x}}}$$

(2-43) $y = \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} - \dfrac{1}{2}x\sqrt{a^2-x^2}$

$$y' = \frac{a^2}{2}\frac{(\frac{x}{a})'}{\sqrt{1-\frac{x^2}{a^2}}} - \frac{1}{2}\sqrt{a^2-x^2} - \frac{1}{2}x\frac{1}{2}\frac{-2x}{\sqrt{a^2-x^2}}$$
$$= \frac{a^2}{2\sqrt{a^2-x^2}} - \frac{1}{2}\sqrt{a^2-x^2} + \frac{x^2}{2\sqrt{a^2-x^2}}$$
$$= \frac{a^2-(a^2-x^2)+x^2}{2\sqrt{a^2-x^2}}$$
$$= \boxed{\frac{x^2}{\sqrt{a^2-x^2}}}$$

(2-44) $y = \dfrac{1}{\sqrt{3}}\log\left|\dfrac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|$

$$y' = \frac{1}{\sqrt{3}}\left[\left(\log\left|\sqrt{3}+\tan\frac{x}{2}\right|\right)' - \left(\log\left|\sqrt{3}-\tan\frac{x}{2}\right|\right)'\right]$$
$$= \frac{1}{\sqrt{3}}\left[\frac{\frac{1}{\cos^2\frac{x}{2}}\frac{1}{2}}{\sqrt{3}+\tan\frac{x}{2}} - \frac{-\frac{1}{\cos^2\frac{x}{2}}\frac{1}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right]$$
$$= \frac{1}{2\sqrt{3}\cos^2\frac{x}{2}}\left[\frac{1}{\sqrt{3}+\tan\frac{x}{2}} + \frac{1}{\sqrt{3}-\tan\frac{x}{2}}\right]$$
$$= \frac{1}{2\sqrt{3}\cos^2\frac{x}{2}}\frac{2\sqrt{3}}{3-\tan^2\frac{x}{2}}$$
$$= \frac{1}{3\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$$
$$= \frac{2}{3(1+\cos x)-(1-\cos x)}$$
$$= \boxed{\frac{1}{1+2\cos x}}$$

(2-45) $y = \dfrac{1}{4\sqrt{2}}\log\left|\dfrac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| + \dfrac{1}{2\sqrt{2}}\left(\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1)\right)$

$$y' = \frac{1}{4\sqrt{2}}\left(\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}}{1+(\sqrt{2}x+1)^2} + \frac{\sqrt{2}}{1+(\sqrt{2}x-1)^2}\right)$$
$$= \frac{1}{4}\left(\frac{\sqrt{2}x+1}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{4}\left(\frac{1}{x^2+\sqrt{2}x+1} + \frac{1}{x^2-\sqrt{2}x+1}\right)$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}\right) \\
&= \frac{1}{4}\frac{(\sqrt{2}x+2)(x^2-\sqrt{2}x+1)-(\sqrt{2}x-2)(x^2+\sqrt{2}x+1)}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} \\
&= \boxed{\frac{1}{1+x^4}}
\end{aligned}
$$