数学演習 I　第 10 回　積分法 I

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問題 1 以下の不定積分を計算せよ.

(1-1) $\int (-3)\mathrm{d}x = \boxed{-3x + C}$

(1-2) $\int (2x-1)\mathrm{d}x = \boxed{x^2 - x + C}$

(1-3) $\int (1-2x+3x^2)\mathrm{d}x = \boxed{x - x^2 + x^3 + C}$

(1-4) $\int 4x^7\mathrm{d}x = \boxed{\frac{1}{2}x^8 + C}$

(1-5) $\int (x+3)(x-3)\mathrm{d}x \int (x^2-9)\mathrm{d}x = \boxed{\frac{1}{3}x^3 - 9x + C}$

(1-6) $\int (2x+1)(3x-2)\mathrm{d}x = \int (6x^2-x-2)\mathrm{d}x = \boxed{2x^3 - \frac{1}{2}x^2 - 2x + C}$

問題 2 以下の条件を満たす関数 $f(x)$ を求めよ.

(2-1) $f(0) = 1,\ f'(x) = 3x^2 + 2x + 5$

$$f(x) = \int (3x^2+2x+5)\mathrm{d}x = x^3 + x^2 + 5x + C$$

$f(0) = 1$ より，$C = 1$ なので，$f(x) = \boxed{x^3 + x^2 + 5x + 1}$

(2-2) $f(3) = -1,\ f'(x) = (x+2)(x-1)$

$$f(x) = \int (x+2)(x-1)\mathrm{d}x = \int (x^2+x-2)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C$$

$f(3) = -1$ より，$-1 = 9 + \frac{9}{2} - 6 + C = \frac{15}{2} + C$ なので，$C = -\frac{17}{2}$ より，

$f(x) = \boxed{\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x - \frac{17}{2}}$

問題 3 以下の定積分を計算せよ.

(3-1) $\int_1^2 (-5)\mathrm{d}x = -5[x]_1^2 = (-5)(2-1) = \boxed{-5}$

(3-2) $\int_0^1 (4x-3)\mathrm{d}x = [2x^2 - 3x]_0^1 = \boxed{-1}$

(3-3) $\int_{-1}^1 x^3\mathrm{d}x$

奇関数 $(f(-x) = -f(x))$ なので $\boxed{0}$

(3-4) $\int_{-1}^{1} x^4 \mathrm{d}x$

偶関数 $(f(-x) = f(x))$ なので $\int_{-1}^{1} x^4 \mathrm{d}x = 2\int_{0}^{1} x^4 \mathrm{d}x = 2\left[\frac{1}{5}x^5\right]_0^1 = \boxed{\frac{2}{5}}$

(3-5) $\int_{-1}^{1} (x^3 + x^4)\mathrm{d}x = 2\int_{0}^{1} x^4 \mathrm{d}x = 2\left[\frac{1}{5}x^5\right]_0^1 = \boxed{\frac{2}{5}}$

(3-6) $\int_{1}^{2} (x^2 - x)\mathrm{d}x = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_1^2 = \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \boxed{\frac{5}{6}}$

(別解)(理解できなければ無視して可)

$\int_{1}^{2} (x^2 - x)\mathrm{d}x = \int_{1}^{2} x(x-1)\mathrm{d}x = \int_{0}^{1} y(y+1)\mathrm{d}x = \left[\frac{1}{3}y^3 + \frac{1}{2}y^2\right]_0^1 = \boxed{\frac{5}{6}}$

(3-7) $\int_{0}^{1} (2x-1)^2\mathrm{d}x = \int_{0}^{1} (4x^2 - 4x + 1)\mathrm{d}x = \left[\frac{4}{3}x^3 - 2x^2 + x\right]_0^1 = \boxed{\frac{1}{3}}$

(別解)(理解できなければ無視して可)

$\int_{0}^{1} (2x-1)^2\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} y^2\mathrm{d}y = \int_{0}^{1} y^2\mathrm{d}y = \boxed{\frac{1}{3}}$

(3-8) $\int_{-1}^{1} \left(x + \frac{1}{2}\right)^2 \mathrm{d}x = \int_{-1}^{1} \left(x^2 + x + \frac{1}{4}\right) \mathrm{d}x = 2\int_{0}^{1} \left(x^2 + \frac{1}{4}\right) \mathrm{d}x = 2\left[\frac{1}{3}x^3 + \frac{1}{4}x\right]_0^1 = \boxed{\frac{7}{6}}$

(別解)

$\int_{-1}^{1} \left(x + \frac{1}{2}\right)^2 \mathrm{d}x = \left[\frac{1}{3}\left(x + \frac{1}{2}\right)^3\right]_{-1}^1 = \boxed{\frac{7}{6}}$

(3-9) $\int_{2}^{3} (2x+1)(x-1)\mathrm{d}x = \int_{2}^{3} (2x^2 - x - 1)\mathrm{d}x = \left[\frac{2}{3}x^3 - \frac{1}{2}x^2 - x\right]_2^3 = \boxed{\frac{55}{6}}$

(3-10) $\int_{0}^{a} (x+1)^2\mathrm{d}x = \left[\frac{1}{3}(x+1)^3\right]_0^a = \frac{1}{3}\left\{(a+1)^3 - 1\right\} = \boxed{a\left(\frac{a^2}{3} + a + 1\right)}$

(3-11) $\int_{b}^{0} \left(x - \frac{1}{3}\right)^2 \mathrm{d}x = \left[\frac{1}{3}\left(x - \frac{1}{3}\right)^3\right]_b^0 = \frac{1}{3}\left\{\left(-\frac{1}{3}\right)^3 - \left(b - \frac{1}{3}\right)^3\right\} = \boxed{-\frac{1}{3}b^3 + \frac{1}{3}b^2 - \frac{1}{9}b}$

(3-12) $\int_{a}^{b} (4x+1)\mathrm{d}x = \left[2x^2 + x\right]_a^b = (2a^2 + a) - (2b^2 + b) = \boxed{-\frac{1}{3}b^3 + \frac{1}{3}b^2 - \frac{1}{9}b}$

問題 4 以下の定積分を計算せよ.

(4-1) $\int_{0}^{2} x^3\mathrm{d}x + \int_{2}^{4} x^3\mathrm{d}x = \int_{0}^{4} x^3\mathrm{d}x = \left[\frac{1}{4}x^4\right]_0^4 = \boxed{64}$

(4-2) $\int_{-2}^{4} 2x^2\mathrm{d}x + \int_{5}^{-2} 2x^2\mathrm{d}x = \int_{5}^{4} 2x^2\mathrm{d}x = \left[\frac{2}{3}x^3\right]_5^4 = \boxed{-\frac{122}{3}}$

(4-3) $\int_{0}^{1} (x^4 + 2x^2 + 4)\mathrm{d}x + \int_{1}^{0} (x^4 + 2x^2 + 4)\mathrm{d}x = \boxed{0}$

問題 5 以下の 2 つの曲線または直線によって囲まれる領域の面積を求めよ.

(5-1) $y = -(x-4)^2 + 4, \quad x$ 軸

$y=0$ の解は $x=2,6$ なので，

$$S=\int_2^6\left\{-(x-4)^2+4\right\}\mathrm{d}x=\int_{-2}^2\left(-t^2+4\right)\mathrm{d}t=2\int_0^2\left(-t^2+4\right)\mathrm{d}t=2\left[-\frac{1}{3}t^3+4t\right]_0^2$$
$$=\boxed{\frac{32}{3}}$$

(5-2) $y=x^2-x+2,\quad y=-x^2+x+14$
$y_1=x^2-x+2,\quad y_2=-x^2+x+14$ とすると，$y_1-y_2=2(x+2)(x-3)$ より，
$-2<x<3$ において $y_1-y_2<0$ なので，$S=\int_{-2}^3(y_2-y_1)\mathrm{d}x$ となる．

$$S=\int_{-2}^3(-2x^2+2x+12)\mathrm{d}x=\left[-\frac{2}{3}x^3+x^2+12x\right]_{-2}^3=\boxed{\frac{125}{3}}$$

問題 6 不等式 $\displaystyle\int_{\frac{3}{7}}^{-\frac{2}{5}}(x^5+x^3+3x^2-4x+2)\mathrm{d}x>\int_{\frac{3}{7}}^{-\frac{2}{5}}(x^5+x^3+x^2)\mathrm{d}x$ を示せ．

$f(x)=x^5+x^3+3x^2-4x+2,\ g(x)=x^5+x^3+x^2$ とすると，$f(x)-g(x)=2(x-1)^2>0$ より，

$$\int_{\frac{3}{7}}^{-\frac{2}{5}}f(x)\mathrm{d}x-\int_{\frac{3}{7}}^{-\frac{2}{5}}g(x)\mathrm{d}x=\int_{\frac{3}{7}}^{-\frac{2}{5}}\left\{f(x)-g(x)\right\}\mathrm{d}x>0$$

なので，$\displaystyle\int_{\frac{3}{7}}^{-\frac{2}{5}}f(x)\mathrm{d}x>\int_{\frac{3}{7}}^{-\frac{2}{5}}g(x)\mathrm{d}x$ を得る．

問題 7 関数 $f(x)$ が $\displaystyle f(x)=x-\int_0^1 f(x)\mathrm{d}x$ を満たすとき，$f(x)$ を求めよ．

$\displaystyle\int_0^1 f(x)\mathrm{d}x=a$ とおくと，$f(x)=x-a$ より，

$$a=\int_0^1 f(x)\mathrm{d}x=\int_0^1(x-a)\mathrm{d}x=\left[\frac{1}{2}x^2-ax\right]_0^1=\frac{1}{2}-a$$

より，$a=\frac{1}{2}-a$ を解いて $a=\frac{1}{4}$．よって，$\boxed{f(x)=x-\frac{1}{4}}$

問題 8 放物線 $y=x^2-4x+c$ と x 軸によって囲まれる領域の面積が $8\sqrt{6}$ であるとき，
c の値を求めよ．

$x^2-4x+c=0$ の解を $x=\alpha,\beta\ (\alpha>\beta)$ とすると，

$$8\sqrt{6}=\int_\beta^\alpha-(x^2-4x+c)\mathrm{d}x=\left[-\frac{1}{3}x^3+2x^2-cx\right]_\beta^\alpha=-\frac{1}{3}(\alpha^3-\beta^3)+2(\alpha^2-\beta^2)-c(\alpha-\beta)$$

となる．ここで，$x^2-4x+c=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$ の係数を比較して，

$$\alpha+\beta=4,$$
$$\alpha\beta=c$$

なので，$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=4(4-c)$ より，$\alpha-\beta=2\sqrt{4-c}$．
$\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)=8\sqrt{4-c}$．

$\alpha^3-\beta^3=(\alpha-\beta)(\alpha^2+\beta^2+\alpha\beta)=(\alpha-\beta)\left\{(\alpha+\beta)^2-\alpha\beta\right\}=2\sqrt{4-c}(16-c)$.
以上より，

$$8\sqrt{6}=-\frac{2}{3}\sqrt{4-c}(16-c)+16\sqrt{4-c}-2c\sqrt{4-c}=\frac{4}{3}(4-c)\sqrt{4-c},$$
$$(4-c)\sqrt{4-c}=6\sqrt{6},$$
$$(4-c)^{\frac{3}{2}}=6^{\frac{3}{2}},$$
$$4-c=6,$$
$$\boxed{c=-2}$$