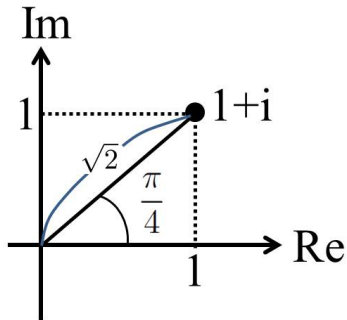
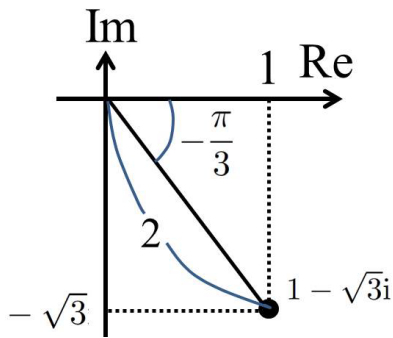


問題 1 以下の複素数を複素平面上に図示し，極形式で表せ。

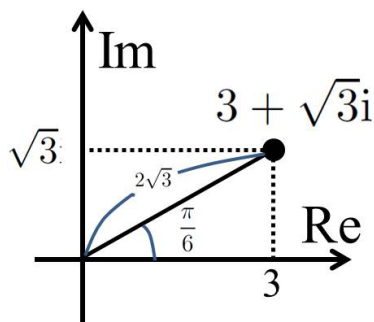
(1-1) $z = 1 + i,$
 $|z| = \sqrt{1^2 + 1^2} = \sqrt{2},$
 $z = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \sqrt{2}e^{i\frac{\pi}{4}}.$



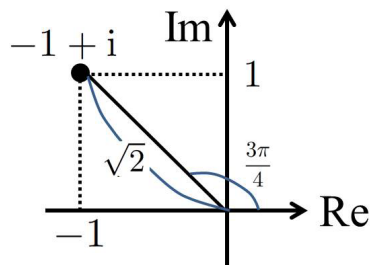
(1-2) $z = 1 - \sqrt{3}i,$
 $|z| = \sqrt{1^2 + (-\sqrt{3})^2} = 2,$
 $z = 2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) = 2 \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right) = 2e^{-i\frac{\pi}{3}}.$



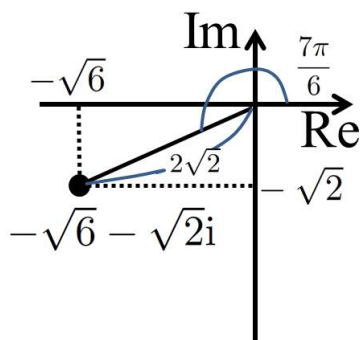
(1-3) $z = 3 + \sqrt{3}i,$
 $|z| = \sqrt{3^2 + \sqrt{3}^2} = 2\sqrt{3},$
 $z = 2\sqrt{3} \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = 2\sqrt{3} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = 2\sqrt{3}e^{i\frac{\pi}{6}}.$



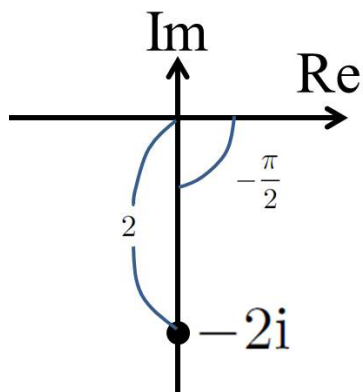
(1-4) $z = -1 + i$,
 $|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$,
 $z = \sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = \sqrt{2}e^{i\frac{3\pi}{4}}$



(1-5) $z = -\sqrt{6} - \sqrt{2}i$,
 $|z| = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2} = 2\sqrt{2}$,
 $z = 2\sqrt{2} \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = 2\sqrt{2} \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) = 2\sqrt{2}e^{i\frac{7\pi}{6}}$

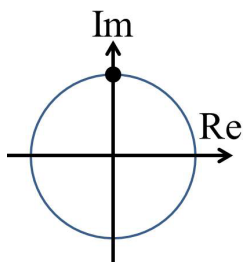


(1-6) $z = -2i$,
 $|z| = \sqrt{0^2 + (-2)^2} = 2$,
 $z = 2(-i) = 2 \left(\cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right) \right) = 2e^{-i\frac{\pi}{2}}$

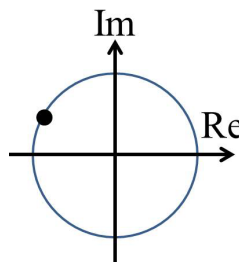


問題 2 以下の複素数を複素平面上に図示し, $x + iy$ (x, y は実数) の形に表せ。

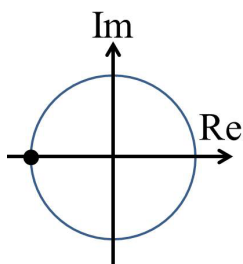
(2-1) $e^{i\frac{\pi}{2}} = i$



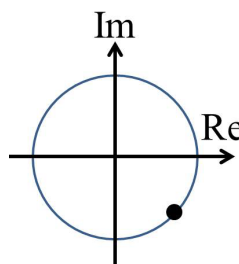
(2-4) $e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$



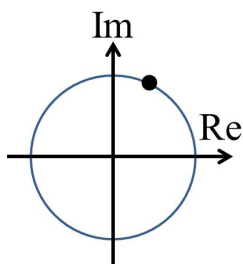
(2-2) $e^{i\pi} = -1$



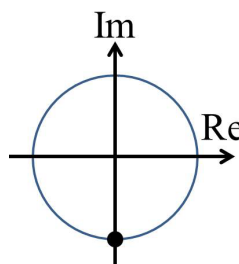
(2-5) $e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$



(2-3) $e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$



(2-6) $e^{-i\frac{\pi}{2}} = -i$



問題 3 $\alpha = 1 + \sqrt{3}i$, $\beta = 1 + i$ とするとき, 以下の値を求めよ。

(3-1) α^6

$\alpha = 2e^{i\frac{\pi}{3}}$ より,

$$\alpha^6 = (2e^{i\frac{\pi}{3}})^6 = 2^6(e^{i\frac{\pi}{3}})^6 = 2^6 e^{i\frac{\pi}{3} \times 6} = 64e^{2\pi i} = 64$$

(3-2) β^8

$\beta = \sqrt{2}e^{i\frac{\pi}{4}}$ より,

$$\beta^8 = (\sqrt{2})^8(e^{i\frac{\pi}{4}})^8 = 2^4(e^{i\frac{\pi}{4}})^8 = 2^4 e^{i\frac{\pi}{4} \times 8} = 16e^{2\pi i} = 16$$

(3-3) $\left(\frac{\alpha}{\beta}\right)^{12}$

$\alpha = 2e^{i\frac{\pi}{3}}$, $\beta = \sqrt{2}e^{i\frac{\pi}{4}}$ より, $\frac{\alpha}{\beta} = \sqrt{2}e^{i(\frac{\pi}{3}-\frac{\pi}{4})} = \sqrt{2}e^{i\frac{\pi}{12}}$ を得る。これより,

$$\left(\frac{\alpha}{\beta}\right)^{12} = (\sqrt{2}e^{i\frac{\pi}{12}})^{12} = (\sqrt{2})^{12} (e^{i\frac{\pi}{12}})^{12} = (\sqrt{2})^{12} e^{i\frac{\pi}{12} \times 12} = 2^6 e^{i\pi} = -64$$

$$(3-4) \left(\frac{\alpha}{\sqrt{2}\beta} \right)^{2014}$$

$$\alpha = 2e^{i\frac{\pi}{3}}, \sqrt{2}\beta = 2e^{i\frac{\pi}{4}} \text{ より, } \frac{\alpha}{\sqrt{2}\beta} = e^{i(\frac{\pi}{3}-\frac{\pi}{4})} = e^{i\frac{\pi}{12}} \text{ なので,}$$

$$\begin{aligned} \left(\frac{\alpha}{\sqrt{2}\beta} \right)^{2014} &= \left(e^{i\frac{\pi}{12}} \right)^{2014} = e^{i\frac{\pi}{12} \times 2014} = e^{i\frac{1007}{6}\pi} = e^{i(168-\frac{1}{6})\pi} \\ &= e^{168i\pi - i\frac{\pi}{6}} = e^{168i\pi} e^{-i\frac{\pi}{6}} = e^{-i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i \end{aligned}$$

を得る。

問題 4 $\alpha = e^{i\pi/3}$, $\beta = e^{i\pi/4}$ とする。 α/β を計算することにより,

$\cos \frac{\pi}{12}$ および $\sin \frac{\pi}{12}$ の値を求めよ。

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{e^{i\frac{\pi}{3}}}{e^{i\frac{\pi}{4}}} \\ &= e^{i(\frac{\pi}{3}-\frac{\pi}{4})} \\ &= e^{i\frac{\pi}{12}} \\ &= \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \end{aligned}$$

一方、

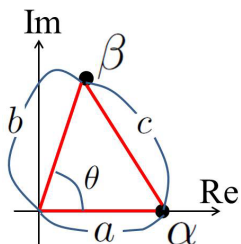
$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{e^{i\frac{\pi}{3}}}{e^{i\frac{\pi}{4}}} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i} \\ &= \frac{1}{\sqrt{2}} \frac{1 + \sqrt{3}i}{1 + i} \\ &= \frac{1}{\sqrt{2}} \frac{(1 + \sqrt{3}i)(1 - i)}{(1 + i)(1 - i)} \\ &= \frac{(\sqrt{3} + 1) + i(\sqrt{3} - 1)}{2\sqrt{2}} \\ &= \frac{(\sqrt{6} + \sqrt{2}) + i(\sqrt{6} - \sqrt{2})}{4} \end{aligned}$$

実部と虚部を比較して、 $\cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$, $\sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ を得る。

問題 5 正の実数 a, b と $0 < \theta < \pi$ に対し, 複素数 α, β を $\alpha = a, \beta = be^{i\theta}$ で定める。

(5-1) 複素数 α, β を複素平面内に図示せよ。

(5-2) $c = |\alpha - \beta|$ とするとき, c を前問で描いた複素平面内に図示せよ。



(5-3) 余弦定理 $c^2 = a^2 + b^2 - 2ab \cos \theta$ を証明せよ。

$$\begin{aligned} c^2 &= |\alpha - \beta|^2 \\ &= |a - be^{i\theta}|^2 \\ &= |(a - b \cos \theta) + i(-b \sin \theta)|^2 \\ &= (a - b \cos \theta)^2 + (-b \sin \theta)^2 \\ &= a^2 + b^2 - 2ab \cos \theta \end{aligned}$$

問題 6 $a = \frac{\sqrt{3}+1}{2}, b = \frac{\sqrt{3}-1}{2}, z = \frac{a+ib}{a-ib} - \frac{a-ib}{a+ib} + 1$ のとき, z^{2014} を求めよ。

$$\begin{aligned} z &= \frac{a+ib}{a-ib} - \frac{a-ib}{a+ib} + 1 \\ &= \frac{(a+ib)^2 - (a-ib)^2}{(a-ib)(a+ib)} + 1 \\ &= \frac{4iab}{a^2 + b^2} + 1 \\ &= \frac{4i \frac{1}{2}}{2} + 1 \\ &= 1 + i \\ &= \sqrt{2}e^{i\frac{\pi}{4}} \end{aligned}$$

より、

$$\begin{aligned} z^{2014} &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{2014} \\ &= \left(\sqrt{2}\right)^{2014} \left(e^{i\frac{\pi}{4}}\right)^{2014} \\ &= 2^{1007} e^{i\frac{\pi}{4} \times 2014} \\ &= 2^{1007} e^{i\pi(504 - \frac{1}{2})} \\ &= 2^{1007} e^{504\pi i} e^{-i\frac{\pi}{2}} \\ &= 2^{1007} \times 1 \times (-i) \\ &= -2^{1007}i. \end{aligned}$$