

問題1 スカラー場  $f$  とベクトル場  $\vec{g}$  が、

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$\vec{g}(x, y, z) = (x + y + z, xy + yz + zx, xyz)$$

で与えられているとき、以下の量を計算せよ。

$$\vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \quad \Delta = \vec{\nabla} \cdot \vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$(1-1) \vec{\nabla} f = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 2y, 2z)$$

$$(1-2) \vec{\nabla} \cdot \vec{g} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (g_x, g_y, g_z) = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} = 1 + x + z + xy$$

$$(1-3) \vec{\nabla} \times \vec{g} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (g_x, g_y, g_z) = \left( \frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}, \frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}, \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \right)$$

$$= (xz - y - x, 1 - yz, y + z - 1)$$

$$(1-4) \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x}(x^2 + y^2 + z^2) \right) = \frac{\partial}{\partial x} 2x = 2.$$

同様に、 $\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 2$  なので、

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + 2 = 6$$

問題2 位置ベクトルを

$$\vec{r} = (x, y, z)$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

と書くとき、以下の量を計算せよ。

$$(2-1) \vec{\nabla} r = \frac{\vec{r}}{r}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2}$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(x^2 + y^2 + z^2)'$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} 2x$$

$$= x(x^2 + y^2 + z^2)^{-1/2}$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x}{r}$$

同様に、

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

以上より、

$$\begin{aligned}\vec{\nabla} r &= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) r \\ &= \left( \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right) \\ &= \left( \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \\ &= \frac{1}{r}(x, y, z) \\ &= \frac{\vec{r}}{r}.\end{aligned}$$

$$(2-2) \quad \vec{\nabla} \cdot \vec{r} = 3$$

$$\begin{aligned}\vec{\nabla} \cdot \vec{r} &= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x, y, z) \\ &= \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z \\ &= 1 + 1 + 1 = 3.\end{aligned}$$

$$(2-3) \quad \vec{\nabla} \times \vec{r} = 0$$

$$\begin{aligned}\vec{\nabla} \times \vec{r} &= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (x, y, z) \\ &= \left( \frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y, \frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z, \frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x \right) \\ &= (0 - 0, 0 - 0, 0 - 0) = 0.\end{aligned}$$

$$(2-4) \quad \vec{\nabla} \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

$$\begin{aligned}\frac{\partial}{\partial x} \frac{1}{r} &= \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-1/2} \\ &= -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x) \\ &= -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} 2x \\ &= -x (x^2 + y^2 + z^2)^{-3/2} \\ &= -\frac{x}{\left( \sqrt{x^2 + y^2 + z^2} \right)^3} \\ &= -\frac{x}{r^3}\end{aligned}$$

同様に、

$$\frac{\partial}{\partial y} \frac{1}{r} = -\frac{y}{r^3}, \quad \frac{\partial}{\partial z} \frac{1}{r} = -\frac{z}{r^3}.$$

以上より、

$$\begin{aligned} \vec{\nabla} \frac{1}{r} &= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{1}{r} \\ &= \left( \frac{\partial}{\partial x} \frac{1}{r}, \frac{\partial}{\partial y} \frac{1}{r}, \frac{\partial}{\partial z} \frac{1}{r} \right) \\ &= \left( -\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3} \right) \\ &= -\frac{1}{r^3} (x, y, z) \\ &= -\frac{\vec{r}}{r^3}. \end{aligned}$$

$$(2-5) \quad \Delta \frac{1}{r} = 0$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \frac{1}{r} &= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \frac{1}{r} \right) \\ &= -\frac{\partial}{\partial x} \frac{x}{r^3} \\ &= -\frac{\partial}{\partial x} x (x^2 + y^2 + z^2)^{-3/2} \\ &= -(x^2 + y^2 + z^2)^{-3/2} + \frac{3}{2} x (x^2 + y^2 + z^2)^{-5/2} 2x \\ &= -\frac{1}{r^3} + \frac{3x^2}{r^5} \end{aligned}$$

同様に、

$$\frac{\partial^2}{\partial y^2} \frac{1}{r} = -\frac{1}{r^3} + \frac{3y^2}{r^5}, \quad \frac{\partial^2}{\partial z^2} \frac{1}{r} = -\frac{1}{r^3} + \frac{3z^2}{r^5}.$$

以上より、

$$\begin{aligned} \Delta \frac{1}{r} &= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{1}{r} \\ &= \frac{\partial^2}{\partial x^2} \frac{1}{r} + \frac{\partial^2}{\partial y^2} \frac{1}{r} + \frac{\partial^2}{\partial z^2} \frac{1}{r} \\ &= -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} \\ &= -\frac{3}{r^3} + \frac{3r^2}{r^5} \\ &= 0. \end{aligned}$$

**問題 3** 原点  $(0, 0, 0)$  に電荷  $q$  があるときの電位は

$$\phi(x, y, z) = \frac{q}{4\pi\epsilon_0 r}$$

で与えられる。ただし、 $r = \sqrt{x^2 + y^2 + z^2}$  である。

(3-1) 電場

$$\vec{E} = -\vec{\nabla}\phi$$

を計算せよ。

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\phi \\ &= -\vec{\nabla}\left(\frac{q}{4\pi\epsilon_0 r}\right) \\ &= -\frac{q}{4\pi\epsilon_0}\left(\vec{\nabla}\frac{1}{r}\right) \\ &= -\frac{q}{4\pi\epsilon_0}\left(-\frac{\vec{r}}{r^3}\right) \\ &= \frac{q}{4\pi\epsilon_0 r^3}\vec{r}.\end{aligned}$$

(3-2)  $\vec{\nabla} \cdot \vec{E} = 0$  を示せ。

$$\vec{\nabla} \cdot \vec{E} = -\Delta\phi = -\frac{q}{4\pi\epsilon_0}\left(\Delta\frac{1}{r}\right) = 0$$

(3-3)  $\vec{\nabla} \times \vec{E} = 0$  を示せ。

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= \vec{\nabla} \times \left(\frac{q}{4\pi\epsilon_0 r^3}\vec{r}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left\{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)\right\} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{\partial}{\partial y}\frac{z}{r^3} - \frac{\partial}{\partial z}\frac{y}{r^3}, \frac{\partial}{\partial z}\frac{x}{r^3} - \frac{\partial}{\partial x}\frac{z}{r^3}, \frac{\partial}{\partial x}\frac{y}{r^3} - \frac{\partial}{\partial y}\frac{x}{r^3}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(z\frac{\partial}{\partial y}\frac{1}{r^3} - y\frac{\partial}{\partial z}\frac{1}{r^3}, x\frac{\partial}{\partial z}\frac{1}{r^3} - z\frac{\partial}{\partial x}\frac{1}{r^3}, y\frac{\partial}{\partial x}\frac{1}{r^3} - x\frac{\partial}{\partial y}\frac{1}{r^3}\right)\end{aligned}$$

ここで、

$$\begin{aligned}\frac{\partial}{\partial x}\frac{1}{r^3} &= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-3/2} \\ &= -\frac{3}{2}(x^2 + y^2 + z^2)^{-5/2}2x \\ &= -\frac{3x}{r^5}\end{aligned}$$

同様に、

$$\frac{\partial^2}{\partial y^2}\frac{1}{r} = -\frac{3y}{r^5}, \quad \frac{\partial^2}{\partial z^2}\frac{1}{r} = -\frac{3z}{r^5}.$$

これを代入すれば直ちに  $\vec{\nabla} \times \vec{E} = 0$  を得る。

別解：

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} \phi) = 0$$

ただし、下の問題(4-1)の結果を使った。

**問題 4** 以下の式を示せ

$$(4-1) \vec{\nabla} \times (\vec{\nabla} f) = 0$$

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} f) &= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ &= \left( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial y} \right), \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial z} \right), \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right) \\ &= \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\ &= 0. \end{aligned}$$

$$(4-2) \vec{\nabla} \cdot (\vec{\nabla} \times \vec{f}) = 0$$

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{f}) &= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \\ &= \left( \frac{\partial^2 f_z}{\partial x \partial y} - \frac{\partial^2 f_y}{\partial x \partial z} \right) + \left( \frac{\partial^2 f_x}{\partial y \partial z} - \frac{\partial^2 f_z}{\partial y \partial x} \right) + \left( \frac{\partial^2 f_y}{\partial z \partial x} - \frac{\partial^2 f_x}{\partial z \partial y} \right) \\ &= 0. \end{aligned}$$

$$(4-3) \vec{\nabla} \times (\vec{\nabla} \times \vec{f}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{f}) - \Delta \vec{f}$$

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{f}) &= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \\ &= \left( \frac{\partial}{\partial y} \left( \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) - \frac{\partial}{\partial z} \left( \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right), \right. \\ &\quad \frac{\partial}{\partial z} \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) - \frac{\partial}{\partial x} \left( \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right), \\ &\quad \left. \frac{\partial}{\partial x} \left( \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) \right) \\ &= \left( \frac{\partial^2 f_y}{\partial y \partial x} - \frac{\partial^2 f_x}{\partial y^2} - \frac{\partial^2 f_x}{\partial z^2} + \frac{\partial^2 f_z}{\partial z \partial x}, \right. \\ &\quad \frac{\partial^2 f_z}{\partial z \partial y} - \frac{\partial^2 f_y}{\partial z^2} - \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_x}{\partial x \partial y}, \\ &\quad \left. \frac{\partial^2 f_x}{\partial x \partial z} - \frac{\partial^2 f_z}{\partial x^2} - \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_y}{\partial y \partial z} \right) \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{\partial^2 f_y}{\partial y \partial x} + \frac{\partial^2 f_z}{\partial z \partial x} + \frac{\partial^2 f_x}{\partial x^2} - \frac{\partial^2 f_x}{\partial x^2} - \frac{\partial^2 f_x}{\partial y^2} - \frac{\partial^2 f_x}{\partial z^2}, \right. \\
&\quad \frac{\partial^2 f_z}{\partial z \partial y} + \frac{\partial^2 f_x}{\partial x \partial y} + \frac{\partial^2 f_y}{\partial y^2} - \frac{\partial^2 f_y}{\partial y^2} - \frac{\partial^2 f_y}{\partial z^2} - \frac{\partial^2 f_y}{\partial x^2}, \\
&\quad \left. \frac{\partial^2 f_x}{\partial x \partial z} + \frac{\partial^2 f_y}{\partial y \partial z} + \frac{\partial^2 f_z}{\partial z^2} - \frac{\partial^2 f_z}{\partial z^2} - \frac{\partial^2 f_z}{\partial x^2} - \frac{\partial^2 f_z}{\partial y^2}, \right) \\
&= \left( \frac{\partial}{\partial x} (\vec{\nabla} \cdot \vec{f}) - \Delta f_x, \frac{\partial}{\partial y} (\vec{\nabla} \cdot \vec{f}) - \Delta f_y, \frac{\partial}{\partial z} (\vec{\nabla} \cdot \vec{f}) - \Delta f_z, \right) \\
&= \left( \frac{\partial}{\partial x} (\vec{\nabla} \cdot \vec{f}), \frac{\partial}{\partial y} (\vec{\nabla} \cdot \vec{f}), \frac{\partial}{\partial z} (\vec{\nabla} \cdot \vec{f}) \right) - (\Delta f_x, \Delta f_y, \Delta f_z) \\
&= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (\vec{\nabla} \cdot \vec{f}) - \Delta (f_x, f_y, f_z) \\
&= \vec{\nabla} (\vec{\nabla} \cdot \vec{f}) - \Delta \vec{f}.
\end{aligned}$$