数学演習I　第11回　関数の微分法

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問題1

(1-1) 積の微分公式 $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$ を示せ。

$$\begin{aligned}\{f(x)g(x)\}' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{\{f(x+h)-f(x)\}g(x+h)+f(x)\{g(x+h)-g(x)\}}{h}\\ &= \left(\lim_{h\to 0}\frac{\{f(x+h)-f(x)\}}{h}\right)\left(\lim_{h\to 0}g(x+h)\right)+f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)g(x)+f(x)g'(x)\end{aligned}$$

(1-2) 商の微分公式 $\left\{\dfrac{f(x)}{g(x)}\right\}' = \dfrac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$ を示せ。

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\frac{1}{g}+f\left(\frac{1}{g}\right)'$$

ここで、

$$\left(\frac{1}{g}\right)' = \left(g^{-1}\right)' = (-1)g^{-2}\cdot g' = -\frac{g'}{g^2}$$

なので、

$$\left(\frac{f}{g}\right)' = f'\frac{1}{g}+f\left(-\frac{g'}{g^2}\right) = \frac{f'g-fg'}{g^2}$$

を得る。

問題2 以下で与えられた関数 y を x で微分し、y' を求めよ。ただし、$a, b, c, \ldots$ は定数とする。

(2-1) $y = (x^2+1)^5(3x+1)^4$

$$\begin{aligned}\left\{(x^2+1)^5(3x+1)^4\right\}' &= \left\{(x^2+1)^5\right\}'(3x+1)^4+(x^2+1)^5\left\{(3x+1)^4\right\}'\\ &= 5(x^2+1)^4(x^2+1)'(3x+1)^4+(x^2+1)^5 4(3x+1)^3(3x+1)'\\ &= 10x(x^2+1)^4(3x+1)^4+12(x^2+1)^5(3x+1)^3.\end{aligned}$$

(2-2) $y = x+\sqrt{x^2+a}$

$$\begin{aligned}\left(x+\sqrt{x^2+a}\right)' &= (x)'+\left(\sqrt{x^2+a}\right)'\\ &= 1+\left\{(x^2+a)^{\frac{1}{2}}\right\}'\\ &= 1+\frac{1}{2}(x^2+a)^{-\frac{1}{2}}(x^2+a)'\\ &= 1+\frac{x}{\sqrt{x^2+a}}.\end{aligned}$$

(2-3) $y = e^{ax}$

$$(e^{ax})' = e^{ax}(ax)' = ae^{ax}$$

(2-4) $y = e^{-x^2}$

$$\left(e^{-x^2}\right)' = e^{-x^2}(-x^2)' = -2xe^{-x^2}$$

(2-5) $y = \tan x$

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \\ &= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \qquad (1) \\ &= 1 + \tan^2 x \qquad (2)\end{aligned}$$

今後、(1), (2) 式の両方を使う。

(2-6) $y = \sin ax \cos bx$

$$\begin{aligned}(\sin ax \cos bx)' &= (\sin ax)' \cos bx + \sin ax\,(\cos bx)' \\ &= \cos ax\,(ax)' \cos bx + \sin ax(-\sin bx)\,(bx)' \\ &= a\cos ax \cos bx - b\sin ax \sin bx.\end{aligned}$$

(2-7) $y = \sin^2 ax$

$$\begin{aligned}\left(\sin^2 ax\right)' &= 2\sin ax\,(\sin ax)' \\ &= 2\sin ax \cos ax\,(ax)' \\ &= 2a\sin ax \cos ax \\ &= a\sin 2ax\end{aligned}$$

(2-8) $y = \sin^{-1} x$

$$\begin{aligned}&x = \sin y, \\ &\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}, \\ &(\sin^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{\sqrt{1-x^2}}.\end{aligned}$$

(2-9) $y = \cos^{-1} x$

$$\begin{aligned}&x = \cos y, \\ &\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}, \\ &(\cos^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = -\frac{1}{\sqrt{1-x^2}}.\end{aligned}$$

(2-10) $y = \tan^{-1} x$

$$\begin{aligned}&x = \tan y, \\ &\frac{\mathrm{d}x}{\mathrm{d}y} = 1 + \tan^2 y = 1 + x^2, \\ &(\tan^{-1} x)' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{1+x^2}.\end{aligned}$$

(2-11) $y = (1+x^2)e^{-x}$

$$\begin{aligned}\left\{(1+x^2)e^{-x}\right\}' &= \left\{(1+x^2)\right\}' e^{-x} + (1+x^2)\left(e^{-x}\right)' \\ &= 2xe^{-x} + (1+x^2)e^{-x}(-1) \\ &= (2x-1-x^2)e^{-x} \\ &= -(x-1)^2 e^{-x}\end{aligned}$$

(2-12) $y = e^{\sin x}$

$$\left(e^{\sin x}\right)' = e^{\sin x}(\sin x)' = \cos x e^{\sin x}$$

(2-13) $y = x^x$

$$\begin{aligned}&y = x^x = e^{\log x^x} = e^{x\log x}, \\ &(x^x)' = (e^{x\log x})' = e^{x\log x}(x\log x)' = x^x\left\{(x)'\log x + x(\log x)'\right\} = (1+\log x)x^x\end{aligned}$$

(2-14) $y = x^{\sin x}$

$$\begin{aligned}&y = x^{\sin x} = e^{\log x^{\sin x}} = e^{\sin x\log x}, \\ &(x^{\sin x})' = (e^{\sin x\log x})' = e^{\sin x\log x}(\sin x\log x)' = x^{\sin x}\left\{(\sin x)'\log x + \sin x(\log x)'\right\} \\ &= \left(\cos x\log x + \frac{\sin x}{x}\right)x^{\sin x}\end{aligned}$$

(2-15) $y = \sqrt{\sin 2x}$

$$\begin{aligned}\left(\sqrt{\sin 2x}\right)' &= \left\{(\sin 2x)^{\frac{1}{2}}\right\}' \\ &= \frac{1}{2}(\sin 2x)^{-\frac{1}{2}}(\sin 2x)' \\ &= \frac{1}{2}(\sin 2x)^{-\frac{1}{2}}(\cos 2x)(2x)' \\ &= \frac{\cos 2x}{\sqrt{\sin 2x}}.\end{aligned}$$

(2-16) $y = \dfrac{x^2-1}{2}\log(1-x) - \dfrac{x}{2} - \dfrac{x^2}{4}$

$$\begin{aligned}\left(\frac{x^2-1}{2}\log(1-x) - \frac{x}{2} - \frac{x^2}{4}\right)' &= \left(\frac{x^2-1}{2}\right)'\log(1-x) + \frac{x^2-1}{2}\left\{\log(1-x)\right\}' - \frac{1}{2} - \frac{x}{2} \\ &= x\log(1-x) + \frac{x^2-1}{2}\frac{(1-x)'}{1-x} - \frac{1}{2} - \frac{x}{2} \\ &= x\log(1-x) + \frac{(x+1)(x-1)}{2}\frac{1}{x-1} - \frac{1}{2} - \frac{x}{2} \\ &= x\log(1-x)\end{aligned}$$

(2-17) $y = e^{ax}\cos bx$

$$\begin{aligned}(e^{ax}\cos bx)' &= (e^{ax})'\cos bx + e^{ax}(\cos bx)' \\ &= e^{ax}(a\cos bx - b\sin bx).\end{aligned}$$

(2-18) $y = (\log x)^3$

$$\begin{aligned}\left\{(\log x)^3\right\}' &= 3(\log x)^2(\log x)' \\ &= \frac{3(\log x)^2}{x}.\end{aligned}$$

(2-19) $y = \dfrac{1}{\sqrt{\log x}}$

$$\begin{aligned}\left(\frac{1}{\sqrt{\log x}}\right)' &= \left\{(\log x)^{-\frac{1}{2}}\right\}' \\ &= -\frac{1}{2}(\log x)^{-\frac{3}{2}}(\log x)' \\ &= -\frac{1}{2x\log x\sqrt{\log x}}.\end{aligned}$$

(2-20) $y = \sin x^3$

$$(\sin x^3)' = \cos x^3 (x^3)' = 3x^2\cos x^3.$$

(2-21) $y = e^{\sin^2 ax}$

$$(e^{\sin^2 ax})' = e^{\sin^2 ax}(\sin^2 ax)' = a\sin 2ax e^{\sin^2 ax}. \qquad \text{(問題 (2-7) を参照)}$$

(2-22) $y = x\log x - x$

$$(x\log x - x)' = (x\log x)' - (x)' = (x)'\log x + x(\log x)' - 1 = \log x + x\frac{1}{x} - 1 = \log x.$$

(2-23) $y = \log\left|\tan\dfrac{x}{2}\right|$

$$\begin{aligned}\left(\log\left|\tan\frac{x}{2}\right|\right)' &= \frac{\left(\tan\frac{x}{2}\right)'}{\tan\frac{x}{2}} \\ &= \frac{\frac{1}{\cos^2\frac{x}{2}}\left(\frac{x}{2}\right)'}{\tan\frac{x}{2}} \\ &= \frac{1}{2\cos^2\frac{x}{2}\tan\frac{x}{2}} \\ &= \frac{1}{2\cos\frac{x}{2}\sin\frac{x}{2}} \\ &= \frac{1}{\sin x}.\end{aligned}$$

(2-24) $y = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right|$

$$\begin{aligned}\left(\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right)' &= \frac{1}{2a}\left(\log|x-a| - \log|x+a|\right)' \\ &= \frac{1}{2a}\left(\frac{(x-a)'}{x-a} - \frac{(x+a)'}{x+a}\right) \\ &= \frac{1}{x^2-a^2}.\end{aligned}$$

(2-25) $y = \dfrac{1}{2}(x\sqrt{x^2+a} + a\log|x+\sqrt{x^2+a}|)$

$y' = \sqrt{x^2+a}$

(2-26) $y = \log\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$y' = \dfrac{1}{x\sqrt{1-x^2}}$

(2-27) $y = \log\left|x+\sqrt{x^2+1}\right|$

$y' = \dfrac{1}{\sqrt{x^2+1}}$

(2-28) $y = \dfrac{1}{3}\tan^3 x + 3\tan x - \dfrac{3}{\tan x} - \dfrac{1}{3\tan^3 x}$

$y' = \dfrac{1}{\cos^4 x \sin^4 x}$

(2-29) $y = \dfrac{1}{2}\log|\cos x + \sin x| + \dfrac{x}{2}$

$y' = \dfrac{1}{1+\tan x}$

(2-30) $y = x\sin^{-1} x + \sqrt{1-x^2}$

$y' = \sin^{-1} x$

(2-31) $y = 2a\tan^{-1}\sqrt{\dfrac{a+x}{a-x}} - \sqrt{a^2-x^2}$

$y' = \sqrt{\dfrac{a+x}{a-x}}$

(2-32) $y = \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} - \dfrac{1}{2}x\sqrt{a^2-x^2}$

$y' = \dfrac{x^2}{\sqrt{a^2-x^2}}$

(2-33) $y = \dfrac{1}{\sqrt{3}}\log\left|\dfrac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|$

$y' = \dfrac{1}{1+2\cos x}$

(2-34) $y = \dfrac{1}{4\sqrt{2}}\log\left|\dfrac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| + \dfrac{1}{2\sqrt{2}}\left(\tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1)\right)$

$y' = \dfrac{1}{1+x^4}$