数学演習I　第7回　複素数の極形式

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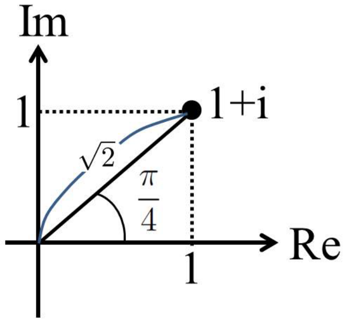
問題1

以下の複素数を複素平面上に図示し、極形式で表せ。

(1-1) $z = 1 + \mathrm{i}$,

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

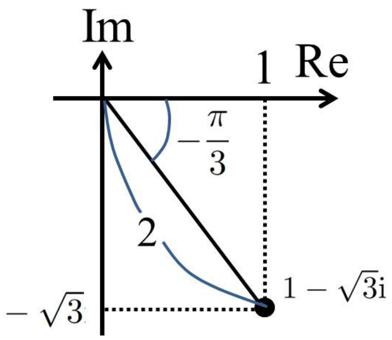
$$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = \sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}.$$



(1-2) $z = 1 - \sqrt{3}\mathrm{i}$,

$$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = 2,$$

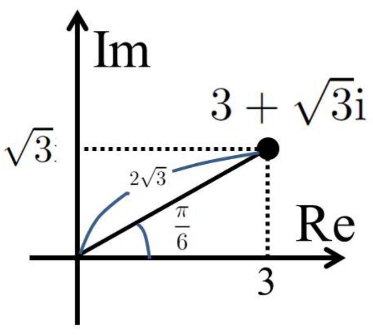
$$z = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = 2\left(\cos(-\frac{\pi}{3}) + \mathrm{i}\sin(-\frac{\pi}{3})\right) = 2e^{-\mathrm{i}\frac{\pi}{3}}.$$



(1-3) $z = 3 + \sqrt{3}\mathrm{i}$,

$$|z| = \sqrt{3^2 + \sqrt{3}^2} = 2\sqrt{3},$$

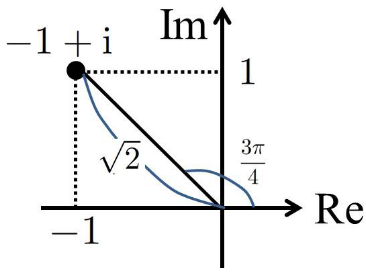
$$z = 2\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = 2\sqrt{3}\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) = 2\sqrt{3}e^{\mathrm{i}\frac{\pi}{6}}.$$



(1-4) $z = -1 + \mathrm{i}$,

$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$,

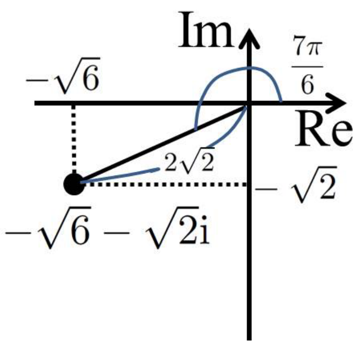
$z = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) = \sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right) = \sqrt{2}e^{\mathrm{i}\frac{3\pi}{4}}$



(1-5) $z = -\sqrt{6} - \sqrt{2}\mathrm{i}$,

$|z| = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2} = 2\sqrt{2}$,

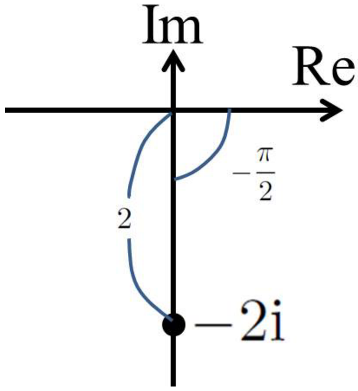
$z = 2\sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) = 2\sqrt{2}\left(\cos\frac{7\pi}{6} + \mathrm{i}\sin\frac{7\pi}{6}\right) = 2\sqrt{2}e^{\mathrm{i}\frac{7\pi}{6}}$



(1-6) $z = -2\mathrm{i}$,

$|z| = \sqrt{0^2 + (-2)^2} = 2$,

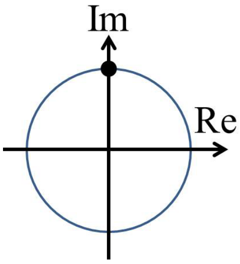
$z = 2(-\mathrm{i}) = 2\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right) = 2e^{-\mathrm{i}\frac{\pi}{2}}$



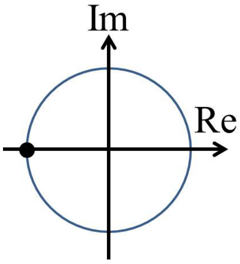
問題 2

以下の複素数を複素平面上に図示し、$x + \mathrm{i}y$, (x, y は実数) の形に表せ。

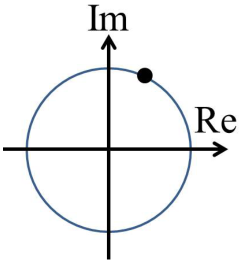
(2-1) $e^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}$



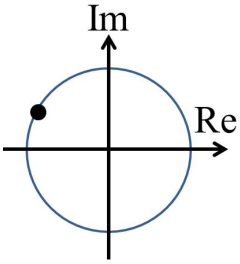
(2-2) $e^{\mathrm{i}\pi} = -1$



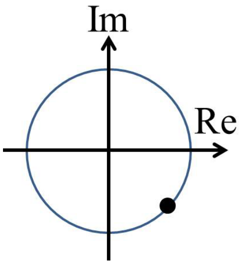
(2-3) $e^{\mathrm{i}\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$



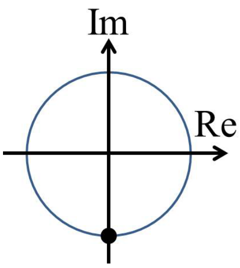
(2-4) $e^{\mathrm{i}\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$



(2-5) $e^{-\mathrm{i}\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}$



(2-6) $e^{-\mathrm{i}\frac{\pi}{2}} = -\mathrm{i}$



問題 3

$\alpha = 1 + \sqrt{3}\mathrm{i},\ \beta = 1 + \mathrm{i}$ とするとき、以下の値を求めよ。

(3-1) α^6

$\alpha = 2e^{\mathrm{i}\frac{\pi}{3}}$ より、

$$\alpha^6 = (2e^{\mathrm{i}\frac{\pi}{3}})^6 = 2^6(e^{\mathrm{i}\frac{\pi}{3}})^6 = 2^6 e^{\mathrm{i}\frac{\pi}{3}\times 6} = 64e^{2\pi\mathrm{i}} = 64$$

(3-2) β^8

$\beta = \sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$ より、

$$\beta^8 = (\sqrt{2})^8(e^{\mathrm{i}\frac{\pi}{4}})^8 = 2^4(e^{\mathrm{i}\frac{\pi}{4}})^8 = 2^4 e^{\mathrm{i}\frac{\pi}{4}\times 8} = 16e^{2\pi\mathrm{i}} = 16$$

(3-3) $\left(\frac{\alpha}{\beta}\right)^{12}$

$\alpha = 2e^{\mathrm{i}\frac{\pi}{3}},\ \beta = \sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$ より、$\frac{\alpha}{\beta} = \sqrt{2}e^{\mathrm{i}(\frac{\pi}{3}-\frac{\pi}{4})} = \sqrt{2}e^{\mathrm{i}\frac{\pi}{12}}$ を得る。これより、

$$\left(\frac{\alpha}{\beta}\right)^{12} = \left(\sqrt{2}e^{\mathrm{i}\frac{\pi}{12}}\right)^{12} = \left(\sqrt{2}\right)^{12}\left(e^{\mathrm{i}\frac{\pi}{12}}\right)^{12} = \left(\sqrt{2}\right)^{12} e^{\mathrm{i}\frac{\pi}{12}\times 12} = 2^6 e^{\mathrm{i}\pi} = -64$$

(3-4) $\left(\frac{\alpha}{\sqrt{2}\beta}\right)^{2014}$

$\alpha = 2e^{\mathrm{i}\frac{\pi}{3}},\ \sqrt{2}\beta = 2e^{\mathrm{i}\frac{\pi}{4}}$ より、$\dfrac{\alpha}{\sqrt{2}\beta} = e^{\mathrm{i}(\frac{\pi}{3}-\frac{\pi}{4})} = e^{\mathrm{i}\frac{\pi}{12}}$ なので、

$$\left(\frac{\alpha}{\sqrt{2}\beta}\right)^{2014} = \left(e^{\mathrm{i}\frac{\pi}{12}}\right)^{2014} = e^{\mathrm{i}\frac{\pi}{12}\times 2014} = e^{\mathrm{i}\frac{1007}{6}\pi} = e^{\mathrm{i}\left(168-\frac{1}{6}\right)\pi}$$
$$= e^{168\mathrm{i}\pi - \mathrm{i}\frac{\pi}{6}} = e^{168\mathrm{i}\pi}e^{-\mathrm{i}\frac{\pi}{6}} = e^{-\mathrm{i}\frac{\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}$$

を得る。

問題 4

$\alpha = e^{\mathrm{i}\pi/3},\ \beta = e^{\mathrm{i}\pi/4}$ とする。α/β を計算することにより、$\cos\dfrac{\pi}{12}$ および $\sin\dfrac{\pi}{12}$ の値を計算せよ。

$$\begin{aligned}\frac{\alpha}{\beta} &= \frac{e^{\mathrm{i}\frac{\pi}{3}}}{e^{\mathrm{i}\frac{\pi}{4}}} \\ &= e^{\mathrm{i}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} \\ &= e^{\mathrm{i}\frac{\pi}{12}} \\ &= \cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\end{aligned}$$

一方、

$$\begin{aligned}\frac{\alpha}{\beta} &= \frac{e^{\mathrm{i}\frac{\pi}{3}}}{e^{\mathrm{i}\frac{\pi}{4}}} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}} \\ &= \frac{1}{\sqrt{2}}\frac{1+\sqrt{3}\mathrm{i}}{1+\mathrm{i}} \\ &= \frac{1}{\sqrt{2}}\frac{(1+\sqrt{3}\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})} \\ &= \frac{(\sqrt{3}+1) + \mathrm{i}(\sqrt{3}-1)}{2\sqrt{2}} \\ &= \frac{(\sqrt{6}+\sqrt{2}) + \mathrm{i}(\sqrt{6}-\sqrt{2})}{4}\end{aligned}$$

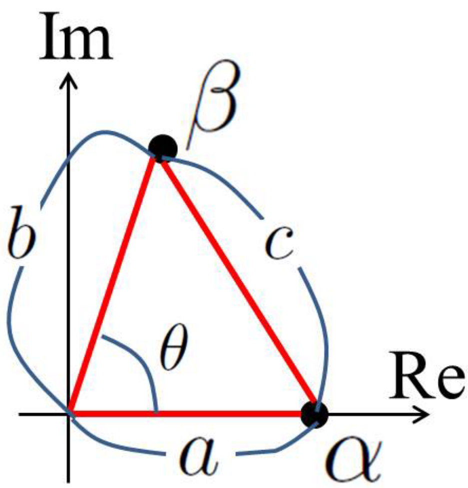
実物と虚部を比較して、$\cos\dfrac{\pi}{12} = \dfrac{\sqrt{6}+\sqrt{2}}{4},\ \sin\dfrac{\pi}{12} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ を得る。

問題 5

正の実数 a, b と $0 < \theta < \pi$ に対し、複素数 α, β を $\alpha = a,\ \beta = be^{\mathrm{i}\theta}$ で定める。

(5-1) 複素数 α, β を複素平面内に図示せよ。

(5-2) $c = |\alpha - \beta|$ とするとき、c を前問で描いた複素平面内に図示せよ。



(5-3) 余弦定理 $c^2 = a^2 + b^2 - 2ab\cos\theta$ を証明せよ。

$$\begin{aligned} c^2 &= |\alpha - \beta|^2 \\ &= |a - be^{\mathrm{i}\theta}|^2 \\ &= |(a - b\cos\theta) + \mathrm{i}(-b\sin\theta)|^2 \\ &= (a - b\cos\theta)^2 + (-b\sin\theta)^2 \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

問題 6

$a = \dfrac{\sqrt{3}+1}{2}$, $b = \dfrac{\sqrt{3}-1}{2}$, $z = \dfrac{a+\mathrm{i}b}{a-\mathrm{i}b} - \dfrac{a-\mathrm{i}b}{a+\mathrm{i}b} + 1$ のとき、z^{2014} を求めよ。

$$\begin{aligned} z &= \frac{a+\mathrm{i}b}{a-\mathrm{i}b} - \frac{a-\mathrm{i}b}{a+\mathrm{i}b} + 1 \\ &= \frac{(a+\mathrm{i}b)^2 - (a-\mathrm{i}b)^2}{(a-\mathrm{i}b)(a+\mathrm{i}b)} + 1 \\ &= \frac{4\mathrm{i}ab}{a^2+b^2} + 1 \\ &= \frac{4\mathrm{i}\,\frac{1}{2}}{2} + 1 \\ &= 1 + \mathrm{i} \\ &= \sqrt{2}e^{\mathrm{i}\frac{\pi}{4}} \end{aligned}$$

より、

$$
\begin{aligned}
z^{2014} &= \left(\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}\right)^{2014} \\
&= \left(\sqrt{2}\right)^{2014}\left(e^{\mathrm{i}\frac{\pi}{4}}\right)^{2014} \\
&= 2^{1007}e^{\mathrm{i}\frac{\pi}{4}\times 2014} \\
&= 2^{1007}e^{\mathrm{i}\pi\left(504-\frac{1}{2}\right)} \\
&= 2^{1007}e^{504\pi\mathrm{i}}e^{-\mathrm{i}\frac{\pi}{2}} \\
&= 2^{1007}\times 1\times(-\mathrm{i}) \\
&= -2^{1007}\mathrm{i}.
\end{aligned}
$$