数学II　第14回　ベクトル解析

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担当：佐藤 純

問題1

スカラー場 f とベクトル場 $\overrightarrow{g}$ が、

$$f(x,y,z) = x^2 + y^2 + z^2,$$
$$\overrightarrow{g}(x,y,z) = (x+y+z, xy+yz+zx, xyz)$$

で与えられているとき、以下の量を計算せよ。

(1-1) $\overrightarrow{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z)$

(1-2) $\overrightarrow{\nabla} \cdot \overrightarrow{g} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (g_x, g_y, g_z) = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} = 1 + x + z + xy$

(1-3) $\overrightarrow{\nabla} \times \overrightarrow{g} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (g_x, g_y, g_z) = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}, \frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}, \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)$
$= (xz - y - x, 1 - yz, y + z - 1)$

(1-4) $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + 2 = 6$

問題2

位置ベクトルを

$$\overrightarrow{r} = (x,y,z)$$
$$r = |\overrightarrow{r}| = \sqrt{x^2+y^2+z^2}$$

と書くとき、以下の量を計算せよ。

(2-1) $\overrightarrow{\nabla} r$

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} \\
&= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(x^2+y^2+z^2)' \\
&= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}2x \\
&= x(x^2+y^2+z^2)^{-1/2} \\
&= \frac{x}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{x}{r}
\end{aligned}$$

同様に、

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

以上より、

$$\begin{aligned}\overrightarrow{\nabla} r &= \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) \\ &= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \\ &= \frac{1}{r}(x, y, z) \\ &= \frac{\overrightarrow{r}}{r}.\end{aligned}$$

(2-2) $\overrightarrow{\nabla} \cdot \overrightarrow{r} = 1 + 1 + 1 = 3$

(2-3) $\overrightarrow{\nabla} \times \overrightarrow{r} = 0$

(2-4) $\overrightarrow{\nabla} \dfrac{1}{r}$

$$\begin{aligned}\frac{\partial}{\partial x}\frac{1}{r} &= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} \\ &= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(x^2 + y^2 + z^2)' \\ &= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} 2x \\ &= -x(x^2 + y^2 + z^2)^{-3/2} \\ &= -\frac{x}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} \\ &= -\frac{x}{r^3}\end{aligned}$$

同様に、

$$\frac{\partial r}{\partial y} = -\frac{y}{r^3}, \quad \frac{\partial r}{\partial z} = -\frac{z}{r^3}.$$

以上より、

$$\begin{aligned}\overrightarrow{\nabla} r &= \left(\frac{\partial}{\partial x}\frac{1}{r}, \frac{\partial}{\partial y}\frac{1}{r}, \frac{\partial}{\partial z}\frac{1}{r}\right) \\ &= \left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right) \\ &= -\frac{1}{r^3}(x, y, z) \\ &= -\frac{\overrightarrow{r}}{r^3}.\end{aligned}$$

(2-5) $\Delta \dfrac{1}{r}$

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}\frac{1}{r} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\frac{1}{r}\right) \\
&= -\frac{\partial}{\partial x}\frac{x}{r^3} \\
&= -\frac{\partial}{\partial x}x(x^2+y^2+z^2)^{-3/2} \\
&= -(x^2+y^2+z^2)^{-3/2} + \frac{3}{2}x(x^2+y^2+z^2)^{-5/2}2x \\
&= -\frac{1}{r^3} + \frac{3x^2}{r^5}
\end{aligned}
$$

同様に、

$$
\frac{\partial^2}{\partial y^2}\frac{1}{r} = -\frac{1}{r^3} + \frac{3y^2}{r^5}, \quad \frac{\partial^2}{\partial z^2}\frac{1}{r} = -\frac{1}{r^3} + \frac{3z^2}{r^5}.
$$

以上より、

$$
\begin{aligned}
\Delta\frac{1}{r} &= \frac{\partial^2}{\partial x^2}\frac{1}{r} + \frac{\partial^2}{\partial y^2}\frac{1}{r} + \frac{\partial^2}{\partial z^2}\frac{1}{r} \\
&= -\frac{3}{r^3} + \frac{3(x^2+y^2+z^2)}{r^5} \\
&= -\frac{3}{r^3} + \frac{3r^2}{r^5} \\
&= 0.
\end{aligned}
$$