数学II　第13回　偏微分

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問題1

2変数 (x,y) の関数 $z(x,y)$ が以下の式で与えられているとき、
一次偏導関数 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ および全微分 $\mathrm{d}z$ を計算せよ。

(1-1) $z = 2x^2y - 3xy^3$

$$\begin{aligned}\frac{\partial z}{\partial x} &= 4xy - 3y^3,\\ \frac{\partial z}{\partial y} &= 2x^2 - 9xy^2,\\ \mathrm{d}z &= (4xy - 3y^3)\mathrm{d}x + (2x^2 - 9xy^2)\mathrm{d}y.\end{aligned}$$

(1-2) $z = \sqrt{x^2 + y^2}$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{x}{z},\\ \frac{\partial z}{\partial y} &= \frac{y}{z},\\ \mathrm{d}z &= \frac{1}{z}(x\mathrm{d}x + y\mathrm{d}y).\end{aligned}$$

(1-3) $z = e^{xy}$

$$\begin{aligned}\frac{\partial z}{\partial x} &= yz,\\ \frac{\partial z}{\partial y} &= xz,\\ \mathrm{d}z &= z(y\mathrm{d}x + x\mathrm{d}y).\end{aligned}$$

(1-4) $z = x\sin y$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \sin y,\\ \frac{\partial z}{\partial y} &= x\cos y,\\ \mathrm{d}z &= \sin y\mathrm{d}x + x\cos y\mathrm{d}y.\end{aligned}$$

(1-5) $z = \log(x^2 + y^2)$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{2x}{x^2 + y^2},\\ \frac{\partial z}{\partial y} &= \frac{2y}{x^2 + y^2},\\ \mathrm{d}z &= \frac{2}{x^2 + y^2}(x\mathrm{d}x + y\mathrm{d}y).\end{aligned}$$

問題 2

2 変数 (x,y) の関数 $z(x,y)$ が以下の式で与えられているとき、

二次偏導関数 $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}, \dfrac{\partial^2 z}{\partial x \partial y}, \dfrac{\partial^2 z}{\partial y \partial x}$ を全て計算し、

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ が成り立つことを確かめよ。

(2-1) $z = 2x^2y - 3xy^3$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}(4xy - 3y^3) = 4y, \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}(2x^2 - 9xy^2) = -18xy, \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}(2x^2 - 9xy^2) = 4x - 9y^2, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}(4xy - 3y^3) = 4x - 9y^2.
\end{aligned}$$

(2-2) $z = e^{xy}$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}(ye^{xy}) = y^2 e^{xy}, \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}(xe^{xy}) = x^2 e^{xy}, \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}(xe^{xy}) = (1 + xy)e^{xy}, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}(ye^{xy}) = (1 + xy)e^{xy}.
\end{aligned}$$

(2-3) $z = \dfrac{x - y}{x + y}$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left\{\frac{2y}{(x+y)^2}\right\} = -\frac{4y}{(x+y)^3}, \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left\{-\frac{2x}{(x+y)^2}\right\} = \frac{4x}{(x+y)^3}, \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left\{-\frac{2x}{(x+y)^2}\right\} = \frac{2(x-y)}{(x+y)^3}, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left\{\frac{2y}{(x+y)^2}\right\} = \frac{2(x-y)}{(x+y)^3}.
\end{aligned}$$

問題 3

(3-1) $z = e^x \sin y,\ x = \cos t,\ y = \sin t$ のとき、$\dfrac{\mathrm{d}z}{\mathrm{d}t}$ を計算せよ。

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= e^x \sin y(-\sin t) + e^x \cos y \cos t \\ &= e^x \cos(y+t).\end{aligned}$$

(3-2) $z = x^2 + y^2,\ x = \dfrac{u}{v},\ y = \dfrac{v}{u}$ のとき、$\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$ を計算せよ。

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= 2x\frac{1}{v} + 2y\left(-\frac{v}{u^2}\right) \\ &= 2\frac{u^4 - v^4}{u^3 v^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 2x\left(-\frac{u}{v^2}\right) + 2y\frac{1}{u} \\ &= 2\frac{v^4 - u^4}{v^3 u^2}.\end{aligned}$$

ここに着任して自由にやっていいって言われて、俺が一番したかったのは、当時ここでも書いたけど、やっぱり数学の魅力と美しさを少しでも伝えること。最後の方は結局俺の力不足で公式暗記詰め込み型になってしまって非常に残念だ。でもだいたい満足してる。

問題 4

$x = 0.1,\ y = 0.2$ のとき、$z = e^{xy}\sin(x+y)$ の値を小数点以下第 3 位まで求めよ。

$$\begin{aligned}e^t &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots, \\ \sin\theta &= \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\end{aligned}$$

より、

$$\begin{aligned}z &= e^{xy}\sin(x+y) \\ &\simeq (1+xy)\left\{(x+y) - \frac{1}{3!}(x+y)^3\right\} \\ &\simeq (x+y) + xy(x+y) - \frac{1}{3!}(x+y)^3 \\ &= 0.3015.\end{aligned}$$

（実際の値は、$z = 0.3014901108408824\cdots$）